Exercise Sheet 1

Affine and Projective Varieties

General Rule: We recommend that you do, or at least try to, solve all unmarked problems. Problems marked $*$ are additional ones that may be harder or may lead into directions not immediately covered by the course. Problems marked $**$ are challenge problems; if you try them, discuss your results with Prof. Pink.

Exercises 1 to 4 are taken or adapted from the book *Algebraic Geometry* by Hartshorne.

Let K be an algebraically closed field.

1. Identifying affine 2-space K^2 with $K^1 \times K^1$ in the natural way, show that the Zariski topology on K^2 is not the product topology of the Zariski topologies on the two copies of K^1.

2. Fix a topological space X. By a subspace of X we always mean a subset with the induced topology. Prove:

   (a) For any irreducible subspace Y the closure $\bar{Y}$ is also irreducible.

   (b) For any subspace Y we have $\dim Y \leqslant \dim X$.

   (c) If $X = \bigcup_{i \in I} U_i$ for open subspaces U_i, then $\dim X = \sup_{i \in I} \dim U_i$.

   (d) Give an example of a noetherian topological space of infinite dimension.

3. Let $R := K[X_0, \ldots, X_n]$. For a homogeneous ideal $\mathfrak{a} \subset R_+$, show that the following conditions are equivalent:

   (a) The zero locus $\bar{V}(\mathfrak{a})$ within $\mathbb{P}^n(K)$ is empty.

   (b) $\operatorname{Rad} \mathfrak{a}$ contains the ideal $R_+ := \boxplus_{d>0} R_d$.

   (c) $\mathfrak{a}$ contains R_d for some $d > 0$.

4. (*d-uple embedding*, or *Veronese map*) For given $n, d > 0$, let $M_0, \ldots, M_N$ be all the monomials of degree d in the $n+1$ variables $X_0, \ldots, X_n$, where $N = \binom{n+d}{d} - 1$.

   (a) Show that there is a well-defined injective map $\rho_d \colon \mathbb{P}^n(K) \to \mathbb{P}^N(K)$ sending $P = (a_0 : \ldots : a_n)$ to $\rho_d(P) := (M_0(a_0, \ldots, a_n) : \ldots : M_N(a_0, \ldots, a_n))$.

   (b) Show that the image of ρ_d is a Zariski closed subvariety defined by equations of the form $Y_i Y_j = Y_k Y_\ell$ for certain tuples (i, j, k, ℓ) of integers in $\{0, \ldots, N\}$.

   (c) Write down the image and the equations explicitly in the cases $(n, d) = (1, 2)$ and $(1, 3)$.

*5. (*Discriminant locus*) To each point $a = (a_0 : \ldots : a_n) \in \mathbb{P}^n(\mathbb{C})$ associate the non-zero homogeneous polynomial

$$f_a \;:=\; a_0 S^n + a_1 S^{n-1} T + \ldots + a_n T^n \;\in\; \mathbb{C}[S,T],$$

which is well-defined up to a factor in $\mathbb{C}^\times$. Let $D \subset \mathbb{P}^n(\mathbb{C})$ denote the set of points a for which $V(f_a)$ has cardinality less than n. Prove that D is a projective variety and find equations for it.

6. Determine the closure of $V(XY - ZT) \subset K^4$ within $\mathbb{P}^4(K)$ and its singular points.