D-MATH
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Algebraic Geometry

FS 2017

Exercise Sheet 10

Schemes over Fields

Exercise 1 is adapted from *Algebraic Geometry I* by Görtz and Wedhorn.

1. Let R be a discrete valuation ring with field of fractions K and residue field k. The natural ring homomorphism $R \to K \times k$ induces a morphism $f\colon X := \operatorname{Spec}(K \times k) \to Y := \operatorname{Spec} R$. Prove that $K \times k$ is a finitely generated R-algebra, that f is bijective, and that $\dim(X) = 0$ and $\dim(Y) = 1$.

2. Show that for any noetherian affine scheme $X = \operatorname{Spec} R$ and any irreducible closed subscheme $Y \subset X$ of codimension r, there exist elements $f_1, \ldots, f_r \in R$ such that Y is an irreducible component of the zero locus $V(f_1, \ldots, f_r)$.

3. Consider field extensions K and L of k. Prove:

   (a) If K/k is algebraic and purely inseparable, then $L \otimes_k K$ possesses a unique prime ideal.

   (b) If K/k is finite separable, then $L \otimes_k K$ is a finite direct sum of fields.

   *(c) If $\bar{k} \otimes_k K$ is reduced for an algebraic closure $\bar{k}$ of k (then K/k is called *separable*), then $L \otimes_k K$ is reduced.

   (d) If K/k possesses a transcendence basis B such that $K/k(B)$ is separable, then $L \otimes_k K$ is reduced and hence K/k separable.

   *(e) If every element of K that is separable algebraic over k already lies in k (then k is called *separably closed in* K), then $L \otimes_k K$ possesses a unique smallest prime ideal.

   (f) If K/k is separable as in (c) and k is separably closed in K as in (e), then $L \otimes_k K$ is an integral domain.

*4. Write out the proof of one of Propositions 5.49–53 of [Görtz-Wedhorn] in all details.