D-MATH
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Algebraic Geometry

FS 2017

Exercise Sheet 13

Rational Maps, Blowups

Exercises 2 to 4 are adapted from *Algebraic Geometry* by Hartshorne.

**1. Consider schemes X and Y over a noetherian scheme S, such that Y is locally of finite type over S. Take a point $x \in X$, let $i\colon \operatorname{Spec}\mathcal{O}_{X,x} \to X$ be the canonical morphism, and consider a morphism $f\colon \operatorname{Spec}\mathcal{O}_{X,x} \to Y$ over S. Prove:

(a) There exists an open neighborhood $U \subset X$ of x and a morphism $\tilde{f}\colon U \to Y$ over S such that $\tilde{f} \circ i = f$.

(b) For any two pairs $(U, \tilde{f})$ and $(U', \tilde{f}')$ as in (a) there exists an open neighborhood $V \subset U \cap U'$ of x such that $\tilde{f}|V = \tilde{f}'|V$.

(c) Do the same conclusions hold without the assumption that S is noetherian? If not, which other condition can one put in its place?

2. Show that any integral scheme of finite type over a field k and of dimension r is birational to a hypersurface in $\mathbb{P}_k^{r+1}$.

3. An integral scheme X of finite type over a field k is called *rational* if it is birationally equivalent to $\mathbb{P}_k^n$ for some n. Prove:

(a) Any irreducible conic $C \subset \mathbb{P}_k^2$ with $C(k) \neq \varnothing$ is a rational curve.

(b) Let X be the nodal cubic curve $V(Y^2Z - X^2(X+Z))$ in $\mathbb{P}_k^2$. Show that the projection from the point $p = (0:0:1)$ to the line $Z = 0$ induces a birational map $\varphi\colon X \dashrightarrow \mathbb{P}_k^1$. Thus X is a rational curve. What are the maximal domains of definition of φ and φ^{-1}?

4. A birational map $\varphi\colon \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^2$ over a field k is called a *plane Cremona transformation.* We give an example, called a *quadratic transformation*, given by $(a_0 : a_1 : a_2) \mapsto (a_1a_2 : a_0a_2 : a_0a_1)$ when no two of a_0, a_1, a_2 are zero.

(a) Show that φ is birational dominant and its own inverse.

(b) Find open subsets $U, V \subset \mathbb{P}^2$ such that $\varphi\colon U \to V$ is an isomorphism.

(c) Find the maximal open subset where φ is defined and the morphism on it.

*5. Let F be the surface obtained from $\mathbb{P}_k^2$ by blowing up the three points $(1:0:0)$, $(0:1:0)$, and $(0:0:1)$. Show that the plane Cremona transformation from the preceding exercise extends to an isomorphism $F \xrightarrow{\sim} F$.

(To construct F, identify each standard chart $D_{X_i} \subset \mathbb{P}_k^2$ with $\mathbb{A}_k^2$ such that the respective given point corresponds to the origin $0 \in \mathbb{A}_k^2$, construct $\tilde{D}_{X_i} \to D_{X_i}$ as in the blowup of $\mathbb{A}_k^2$ at the origin, then glue the $\tilde{D}_{X_i}$ to a scheme F over $\mathbb{P}_k^2$.)