Algebraic Geometry

FS 2017

Exercise Sheet 14

Blowups, Curves

Exercise 3 is taken from *Algebraic Geometry I* by Görtz and Wedhorn. Exercises 4 and 5 are adapted from *Algebraic Geometry* by Hartshorne.

1. Consider any coprime integers $p, q \geqslant 1$. Compute the strict transform of the affine curve $C_{p,q}\colon X^p + Y^q = 0$ under the blowup of $\mathbb{A}^2_k$ in the origin. Deduce that a finite number of iterated blowups makes this curve regular, but the number of blowups needed may be arbitrarily large.

2. Determine the strict transform of the following surface when blowing up the origin. Repeat the procedure in suitable local coordinates until no singular points are left.

   (a) $V(X^2 + Y^2 + Z^3)$

   *(b) (for masochists) $V(X^2 + Y^3 + Z^5)$

3. Let C be an integral curve over a field k. Show that C is proper over k if and only if its normalization $\tilde{C}$ is proper over k.

4. Let k be an algebraically closed field. Let C be a regular integral curve that is separated of finite type over k which is birational to, but not isomorphic to $\mathbb{P}^1_k$.

   (a) Show that C is isomorphic to an open subset of $\mathbb{A}^1_k$.

   (b) Show that C is affine.

   (c) Show that the affine coordinate ring $\mathcal{O}_C(C)$ is a unique factorization domain.

*5. Let k be an algebraically closed field of characteristic $\neq 2$. Let C be the curve $V(Y^2 - X^3 + X) \subset \mathbb{A}^2_k$. In this exercise we will show that C is not birational to $\mathbb{P}^1_k$ over k, hence its function field $K(C)$ is not a pure transcendental extension of k.

   (a) Show that C is nonsingular, and deduce that its coordinate ring $A := k[X,Y]/(Y^2 - X^3 + X)$ is an integrally closed domain.

   (b) Let $k[x]$ be the subring of $K := K(C)$ generated by the image of X in A. Show that $k[x]$ is a polynomial ring, and that A is the integral closure of $k[x]$ in K.

   (c) Show that there is an automorphism $\sigma\colon A \to A$ which sends y to $-y$ and leaves x fixed. For any $a \in A$, define the *norm* of a to be $N(a) := a \cdot \sigma(a)$. Show that $N(a) \in k[x]$ and $N(1) = 1$ and $N(ab) = N(a) \cdot N(b)$ for any $a, b \in A$.

(d) Using the norm, show that the units in A are precisely the nonzero elements of k. Show that x and y are irreducible elements of A. Deduce that A is not a unique factorization domain.

(e) Use the previous exercise to prove that C is not birational to $\mathbb{P}^1_k$ over k.