Exercise Sheet 2

Regular versus Singular Points, Categories

Let K be an algebraically closed field.

1. Consider the standard open affine covering $\mathbb{P}^n(K) = U_0 \cup \ldots \cup U_n$ with the maps $\iota_i \colon K^n \xrightarrow{\sim} U_i$. Let x be a point on a projective variety $X \subset \mathbb{P}^n(K)$. For each $0 \leqslant i \leqslant n$ with $x \in U_i$ let $R_{X,x,i}$ denote the local ring of $\iota_i^{-1}(X \cap U_i)$ at the point $\iota_i^{-1}(x)$. Show that for different i the rings $R_{X,x,i}$ are naturally isomorphic.

2. Let K be a field of characteristic 0. Determine the singular points and the dimensions of the Zariski cotangent spaces at these points for

   (a) $V(I) \subset \mathbb{A}^3(K)$, where $I = (X^3 - Z, Y^2 - Z) \subset K[X,Y,Z]$,

   (b) $\bar{V}(J) \subset \mathbb{P}^3$, where $J = (YZ^2 - W^3) \subset K[X,Y,Z,W]$.

*3. Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k := R/\mathfrak{m}$. Prove that the following are equivalent.

   (a) R is regular of dimension d.

   (b) $\forall n \geqslant 0\colon \dim_k \mathfrak{m}^n/\mathfrak{m}^{n+1} = \binom{n+d-1}{d-1}$.

   (c) $\operatorname{gr} R := \bigoplus_{n \geqslant 0} \mathfrak{m}^n/\mathfrak{m}^{n+1} \cong k[X_1, \ldots, X_d]$.

   (d) if $R \cong k \oplus \mathfrak{m}$: $\hat{R} \cong k[[X_1, \ldots, X_d]]$, where $\hat{R}$ is the $\mathfrak{m}$-adic completion of R.

4. A morphism $f \colon X \to Y$ in a category $\mathcal{C}$ is called a *monomorphism* if for any object Z in $\mathcal{C}$ the map $\operatorname{Mor}_{\mathcal{C}}(Z,X) \to \operatorname{Mor}_{\mathcal{C}}(Z,Y)$, $g \mapsto f \circ g$ is injective. A morphism is called an *epimorphism* if the corresponding morphism in the opposite category is a monomorphism. For which of the categories **Sets**, **Groups**, R-**Mod**, **Rings**, **Top**, is it true that a morphism is ...

   (a) ... a monomorphism if and only if its underlying map is injective?

   (b) ... an epimorphism if and only if its underlying map is surjective?

   (c) ... an isomorphism if and only if it is both a mono- and an epimorphism?

5. Let $\mathcal{C}$ be a category and $f, g \in \mathrm{Mor}_{\mathcal{C}}(X, X')$. The *difference kernel* of f and g is the limit $\varprojlim(X \overset{f}{\underset{g}{\rightrightarrows}} X')$, if it exists.

   (a) Explain the connection with the kernel when $\mathcal{C}$ is the category of R-modules. (Whence the name!)

   (b) Define the dual notion of *difference cokernel* and explain its relation with the cokernel of an R-module homomorphism.

6. Express the following as a limit or colimit.

   (a) A quotient of a topological space X by a subspace Y.

   (b) The graph of a map of sets $f\colon X \to Y$.

   (c) A module over a ring in terms of its finitely generated submodules.

   (d) The fiber $f^{-1}(y)$ of a map of sets $f\colon X \to Y$.

   (e) The quotient of a set X by an equivalence relation $E \subset X \times X$.

   (f) The equivalence relation $E \subset X \times X$ generated by a relation $R \subset X \times X$.