D-MATH | Algebraic Geometry | FS 2017
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Exercise Sheet 3

Equivalence of Categories, Representable and Adjoint Functors

1. Give an example of a functor that does not preserve monomorphisms. Dito for epimorphisms.

2. Consider a functor $F\colon \mathcal{C} \to \mathcal{D}$. The functor is called *fully faithful* if for all $X, X' \in \mathrm{Ob}(\mathcal{C})$ the map $F_{X,X'}\colon \mathrm{Mor}_{\mathcal{C}}(X,X') \to \mathrm{Mor}_{\mathcal{D}}(FX,FX')$ is bijective. It is called *essentially surjective* if for any $Y \in \mathrm{Ob}(\mathcal{D})$ there exists $X \in \mathrm{Ob}(\mathcal{C})$ such that $FX \cong Y$. Prove that F is part of an equivalence of categories if and only if it is fully faithful and essentially surjective.

3. Show that equivalence of categories is an equivalence relation.

4. Show that the category of sets is not equivalent to its opposite category.

   Hint: Play around with initial and final objects and products and coproducts.

*5. Let **Cat** denote the category of small categories with functors as morphisms. Let $F\colon \mathbf{Cat} \to \mathbf{Sets}$ be the functor that sends a small category $\mathcal{C}$ to the set $\mathrm{Mor}(\mathcal{C})$. Prove that F is corepresentable.

6. Show that the tensor product $M \otimes_R N$ corepresents the functor R-$\mathbf{Mod} \to \mathbf{Sets}$, $L \mapsto \mathrm{Bilin}_R(M \times N, L)$.

*7. Fix topological spaces X and Y and consider the functor $\mathbf{Top}^{\mathrm{opp}} \to \mathbf{Sets}$, $Z \mapsto \mathrm{Mor}_{\mathbf{Top}}(Z \times Y, X)$. Under what conditions is this functor representable?

8. Consider the forgetful functor $\mathbf{Top} \to \mathbf{Sets}$ mapping any topological space to the underlying set and any continuous map to the underlying map. Does this functor have a left or right adjoint, and if so, which?