D-MATH | Algebraic Geometry | FS 2017
Prof. Richard Pink

Exercise Sheet 5

Sheaves of Abelian Groups, Locally Ringed Spaces

Exercises 1, 4 and 5(c) are taken or adapted from *Algebraic Geometry* by Hartshorne. Exercises 3(a,b) and 5 are from *Algebraic Geometry I* by Görtz and Wedhorn.

1. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups. Show that $\mathrm{im}(f) \cong \mathcal{F}/\ker(f)$ and $\mathrm{coker}(f) \cong \mathcal{G}/\mathrm{im}(f)$.

2. Let X and Y be topological spaces and $f\colon X \to Y$ a continuous map.

   (a) Show that for any exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ in $\mathbf{Sh}_{\mathbf{Ab}}(X)$ and any open subset $U \subset X$, the sequence $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$ is exact.

   (b) Let $\mathcal{F}$ be a sheaf of abelian groups on X. Show that $\mathscr{H}\!om(\mathcal{F}, -)$ is a left exact covariant functor and that $\mathscr{H}\!om(-, \mathcal{F})$ is a left exact contravariant functor $\mathbf{Sh}_{\mathbf{Ab}}(X) \to \mathbf{Sh}_{\mathbf{Ab}}(X)$.

   (c) Show that for any exact sequence $0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ in $\mathbf{Sh}_{\mathbf{Ab}}(X)$, the sequence $0 \longrightarrow f_*\mathcal{F} \xrightarrow{f_*\varphi} f_*\mathcal{G} \xrightarrow{f_*\psi} f_*\mathcal{H}$ is exact.

   (d) Show that for any sheaf of abelian groups $\mathcal{G}$ on Y and any $x \in X$ there is a natural isomorphism of stalks $\mathcal{G}_{f(x)} \cong (f^{-1}\mathcal{G})_x$.

   (e) Show that f^{-1} is an exact functor from sheaves of abelian groups on Y to sheaves of abelian groups on X.

3. Consider $\mathbb{C}$ with the analytic topology. Denote by $\mathcal{O}_{\mathbb{C}}$ the sheaf of holomorphic functions on $\mathbb{C}$, and let $\mathcal{O}_{\mathbb{C}}^{\times}$ be the sheaf of nowhere vanishing holomorphic functions. Let $\underline{\mathbb{Z}}$ and $\underline{\mathbb{C}}$ denote the constant sheaves on $\mathbb{C}$ with values in $\mathbb{Z}$, and $\mathbb{C}$, respectively.

   (a) Show that $(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$ is a locally ringed space. What are the residue fields $k(z)$ for $z \in \mathbb{C}$?

   (b) Let $D\colon \mathcal{O}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{C}}$ be the morphism of sheaves which, for $U \subset \mathbb{C}$ open, sends $f \in \mathcal{O}_{\mathbb{C}}(U)$ to its derivative $f' \in \mathcal{O}_{\mathbb{C}}(U)$. Show that there is a natural exact sequence of sheaves
   $$0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \mathcal{O}_{\mathbb{C}} \xrightarrow{D} \mathcal{O}_{\mathbb{C}} \longrightarrow 0.$$

   (c) Show that there is a natural exact sequence of sheaves
   $$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_{\mathbb{C}} \xrightarrow{f \mapsto e^{2\pi i f}} \mathcal{O}_{\mathbb{C}}^{\times} \longrightarrow 1.$$

   (d) Which of the sequences in (b) and (c) are exact sequences in the sense of presheaves?

4. Let R be a ring and $X := \operatorname{Spec} R$. Show that for any $a \in R$, the locally ringed space $(D_a, \mathcal{O}_X|_{D_a})$ is isomorphic to $(\operatorname{Spec} R_a, \mathcal{O}_{\operatorname{Spec} R_a})$.

5. Let $(X, \mathcal{O}_X)$ be a locally ringed space.

   (a) Let $U \subset X$ be an open and closed subset. Show that there exists a unique section $e_U \in \mathcal{O}_X(X)$ such that $e_U|_V = 1$ for all open subsets V of U and $e_U|_V = 0$ for all open subsets V of $X \setminus U$. Show that $U \mapsto e_U$ yields a bijection between the set of open and closed subsets of X and the set of idempotent elements of the ring $\mathcal{O}_X(X)$.

   (b) Show that $e_U e_{U'} = e_{U \cap U'}$ for all open and closed subsets $U, U' \subset X$.

   (c) Suppose $X = \operatorname{Spec} R$ for some ring R. Prove that the following are equivalent:

      (i) X is non-empty and not connected.

      (ii) There exist nonzero elements $e_1, e_2 \in R$ such that $e_1 e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 + e_2 = 1$ (these elements are called *orthogonal idempotents).*

      (iii) R is isomorphic to a direct product $R_1 \times R_2$ of nonzero rings.

   (d) Let R be a local ring. Show that $\operatorname{Spec} R$ is connected.