D-MATH
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Algebraic Geometry

FS 2017

Exercise Sheet 7

Subschemes, Finiteness Conditions

Exercises 3 and 9 are taken or adapted from *Algebraic Geometry* by Hartshorne. Exercises 2, 5, 6 and 7 are from *Algebraic Geometry I* by Görtz and Wedhorn.

1. Let k be a field, set $U_i := \mathbb{A}^1_k = \operatorname{Spec} k[X_i]$ for $i = 1, 2$ and consider the open subschemes $U_{ij} := \mathbb{A}^1_k \smallsetminus \{0\} = \operatorname{Spec} k[X_i, X_i^{-1}]$ for $i \neq j$. Let X be the scheme obtained by gluing U_1 and U_2 along U_{12} and U_{21} via $\varphi\colon U_{12} \xrightarrow{\sim} U_{21}$, $X_2 \mapsto X_1$.

   (a) Show that X is not affine.

   (b) Show that X is integral and noetherian.

2. Prove that every locally closed immersion $i\colon Z \to X$ is a monomorphism in the category of schemes.

*3. Let $f\colon Z \to X$ be a morphism of schemes. Show that there is a unique closed subscheme Y of X with the property: the morphism f factors through Y, and if Y' is any other closed subscheme of X through which f factors, then $Y \to X$ also factors through Y'. A reasonable name for this is the *scheme-theoretic closure of the image of f*. Show further that if Z is a reduced scheme, then Y is just the reduced induced structure on the closure of the image $f(Z)$.

**4. Write out the proof of [Görtz-Wedhorn, Theorem 3.42] in all details.

5. Let X be a locally noetherian scheme. Prove that the set of irreducible components of X is locally finite, i.e. that every point of X has an open neighborhood which meets only finitely many irreducible components of X.

6. Let X be a noetherian scheme. Consider the sheaf of ideals $\mathcal{N}_X$ associated to $U \mapsto \operatorname{rad}(\mathcal{O}_X(U))$, the nilradical of X. Show that $\mathcal{N}_X$ is nilpotent, i.e., there exists an integer $k \geqslant 1$ such that $\mathcal{N}_X(U)^k = 0$ for every open subset $U \subset X$.

*7. Let X be a scheme.

   (a) If X is affine, show that X^{red} is affine.

   (b) Assume that X is noetherian. If X^{red} is affine, show that X is affine.
   Hint. Use that $\mathcal{N}_X$ is nilpotent and reduce to the case $\mathcal{N}_X^2 = 0$. Then show that the canonical morphism $X \to \operatorname{Spec}\Gamma(X, \mathcal{O}_X)$ is an isomorphism.

8. Let X be a scheme over a field k. Show that

   (a) if X is locally of finite type over k, then every open covering possesses a refinement to an affine open covering $X = \bigcup_{i \in I}$ such that each $\mathcal{O}_X(U_i)$ is a finitely generated k-algebra.

   (b) X is of finite type over k if and only if X is locally of finite type over k and quasicompact.

9. If X is a scheme of finite type over a field, show that the set of closed points of X is dense in X. Give an example to show that this is not true for arbitrary schemes.