D-MATH | Algebraic Geometry | FS 2017
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Solutions Sheet 1

Affine and Projective Varieties

General Rule: We recommend that you do, or at least try to, solve all unmarked problems. Problems marked $*$ are additional ones that may be harder or may lead into directions not immediately covered by the course. Problems marked $**$ are challenge problems; if you try them, discuss your results with Prof. Pink.

Exercises 1 to 4 are taken or adapted from the book *Algebraic Geometry* by Hartshorne.

Let K be an algebraically closed field.

1. Identifying affine 2-space K^2 with $K^1 \times K^1$ in the natural way, show that the Zariski topology on K^2 is not the product topology of the Zariski topologies on the two copies of K^1.

   Solution: Let x and y be coordinates on K^2. The diagonal in K^2 is the affine algebraic variety $V(x-y)$ and thus a closed set. The Zariski topology on K^1 is the cofinite topology, where the proper closed subsets are precisely the finite subsets. The proper closed subsets in the product topology on $K^1 \times K^1$ are therefore the finite unions of horizontal and/or vertical lines and/or single points, which the diagonal is not.

2. Fix a topological space X. By a subspace of X we always mean a subset with the induced topology. Prove:

   (a) For any irreducible subspace Y the closure $\bar{Y}$ is also irreducible.

   (b) For any subspace Y we have $\dim Y \leqslant \dim X$.

   (c) If $X = \bigcup_{i\in I} U_i$ for open subspaces U_i, then $\dim X = \sup_{i\in I} \dim U_i$.

   (d) Give an example of a noetherian topological space of infinite dimension.

   Solution: (a) Let $Y \subset X$ be irreducible in the subspace topology and suppose $\bar{Y} = Y_1 \cup Y_2$, the union of two closed subsets of X. Then each $Y_i' := Y \cap Y_i$ is closed in the subspace topology on Y, and $Y = Y \cap \bar{Y} = Y_1' \cup Y_2'$. Thus $Y = Y_1'$ or $Y = Y_2'$, meaning that, say, $Y \subset Y_1$. Then $\bar{Y} \subset \bar{Y}_1 = Y_1$ and so $\bar{Y} = Y_1$, as desired.

   (b) Any chain of irreducible closed subsets of Y is of the form $(Y \cap X_0) \subsetneqq \ldots \subsetneqq (Y \cap X_n)$ for closed subsets $X_0, \ldots, X_n$ of X. By (a), the closure $\overline{Y \cap X_i}$ of these sets in X is also irreducible; hence $\overline{Y \cap X_0} \subsetneqq \ldots \subsetneqq \overline{Y \cap X_n}$ is a chain of irreducible closed subsets of X. Varying the chain it follows that $\dim Y \leqslant \dim X$.

   (c) By (b) we have $\dim U_i \leqslant \dim X$ for all i, and thus $\sup_{i\in I} \dim U_i \leqslant \dim X$. Conversely consider some chain $X_0 \subsetneqq \ldots \subsetneqq X_n$ of irreducible closed subsets of X.

Choose i such that $U_i \cap X_0$ is nonempty. Since $X_0 \subsetneqq X_1$, the intersection of U_i with X_1 is nonempty as well. Since X_1 is irreducible and $U_i \cap X_1$ is a nonempty open subset of X_1, this intersection is dense in X_1. Moreover $X_1 \smallsetminus X_0$ is a nonempty open subset of X_1, hence has nonempty intersection with U_i. This implies that $U_i \cap X_0$ is a proper subset of $U_i \cap X_1$. Repeating this process, we obtain a strict chain $U_i \cap X_0 \subsetneqq \ldots \subsetneqq U_i \cap X_n$ of irreducible closed subsets of U_i. It follows that $\dim U_i \geqslant n$. Varying n and i we conclude that $\sup_{i\in I} \dim U_i \geqslant \dim X$.

(d) Endow the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ with the topology where the closed sets are all initial segments. Then any proper closed subset is finite; hence the descending chain condition for closed subsets holds. Also, every non-empty closed subset is irreducible. Thus $\{0\} \subsetneqq \{0, 1\} \subsetneqq \ldots$ is a strictly ascending chain of irreducible closed subsets; hence the dimension is infinite. (The same argument works with any infinite well-ordered set.)

3. Let $R := K[X_0, \ldots, X_n]$. For a homogeneous ideal $\mathfrak{a} \subset R_+$, show that the following conditions are equivalent:

   (a) The zero locus $\bar{V}(\mathfrak{a})$ within $\mathbb{P}^n(K)$ is empty.

   (b) $\operatorname{Rad} \mathfrak{a}$ contains the ideal $R_+ := \boxplus_{d>0} R_d$.

   (c) $\mathfrak{a}$ contains R_d for some $d > 0$.

   Solution (sketch): (a)$\Rightarrow$(b): If $\bar{V}(\mathfrak{a}) = \varnothing$ in $\mathbb{P}^n(K)$, then in K^{n+1} with coordinate ring R we have $V(\mathfrak{a}) \subset \{0\}$. Therefore $\operatorname{Rad}(\mathfrak{a}) \supset I(\{0\}) = R_+$.

   (b)$\Rightarrow$(c): If $\operatorname{Rad} \mathfrak{a}$ contains R_+, then X_i lies in $\operatorname{Rad} \mathfrak{a}$ for each $i \in \{0, \ldots, n\}$. Thus, there exists an $r > 0$ such that X_i^r lies in $\mathfrak{a}$ for each i. Since X_i^r divides any monomial of degree $r(n+1)$, we have $R_{r(n+1)} \subset (X_0^r, \ldots, X_n^r) \subset \mathfrak{a}$.

   (c)$\Rightarrow$(a): Suppose $R_+ \subset \operatorname{Rad} \mathfrak{a}$ and assume $V(\mathfrak{a})$ is nonempty. For any point $P \in V(\mathfrak{a})$ the monomials X_i^d vanish at P for each $0 \leqslant i \leqslant n$ and every $d > 0$. But this is impossible since $P \neq (0 : \ldots : 0)$.

4. (*d-uple embedding*, or *Veronese map*) For given $n, d > 0$, let $M_0, \ldots, M_N$ be all the monomials of degree d in the $n + 1$ variables $X_0, \ldots, X_n$, where $N = \binom{n+d}{d} - 1$.

   (a) Show that there is a well-defined injective map $\rho_d \colon \mathbb{P}^n(K) \to \mathbb{P}^N(K)$ sending $P = (a_0 : \ldots : a_n)$ to $\rho_d(P) := (M_0(a_0, \ldots, a_n) : \ldots : M_N(a_0, \ldots, a_n))$.

   (b) Show that the image of ρ_d is a Zariski closed subvariety defined by equations of the form $Y_iY_j = Y_kY_\ell$ for certain tuples (i, j, k, ℓ) of integers in $\{0, \ldots, N\}$.

   (c) Write down the image and the equations explicitly in the cases $(n, d) = (1, 2)$ and $(1, 3)$.

   Solution (sketch): (a) Write $M_i(P) := M_i(a_0, \ldots, a_n)$. Since the monomials M_i are all homogeneous of degree d, we have $M_i(\lambda P) = \lambda^d M_i(P)$ for each i and any

$\lambda \in K^\times$; hence $\rho_d(\lambda P) = \rho_d(P)$ and so $\rho_d(P)$ is independent of the representative of P. We also need to check that for any $P \in \mathbb{P}^n(K)$ at least one $M_i(P)$ is nonzero. We have $a_k^d = X_k^d(P)$ for each $0 \leqslant k \leqslant n$, and since $P \in \mathbb{P}^n(K)$ we have $X_k^d(P) = a_k^d \neq 0$ for at least one k.

For injectivity, let $Q = (b_0 : \ldots : b_n) \in \mathbb{P}^n(K)$ and suppose $\rho_d(P) = \rho_d(Q)$. We may assume that $a_0^d = X_0^d(P) = 1$. Then $b_0^d = 1$ as well, and without loss of generality $a_0 = b_0 = 1$. If $b_k \neq a_k$ for some index k, then $(X_0^{d-1}X_k)(P) = a_k \neq b_k = (X_0^{d-1}X_k)(Q)$, contradicting the assumption that $\rho_d(P) = \rho_d(Q)$.

(b) Let S be the set of polynomials $Y_iY_j - Y_kY_\ell$ for all tuples (i,j,k,ℓ) for which $M_iM_j = M_kM_\ell$. Clearly, the image of ρ_d is contained in the variety $\bar{V}(S)$. Conversely consider any point $y = (y_0 : \ldots : y_N) \in \bar{V}(S)$. Choose an index i with $y_i \neq 0$ and write $M_i = X_0^{\nu_0} \ldots X_n^{\nu_n}$. Without loss of generality we can assume that $\nu_0 > 0$. For each $1 \leqslant r \leqslant n$ take $0 \leqslant j(r) \leqslant N$ with $M_{j(r)} = M_i \cdot X_r/X_1$, and consider the point $P := (y_i : y_{j(1)} : \ldots : y_{j(n)}) \in \mathbb{P}^n(K)$. Then, using the defining relations in several steps, show that $\rho_d(P) = y$.

(c) For $(n,d) = (1,2)$ we have $N = 2$. Choose coordinates X_0, X_1 on $\mathbb{P}^1(K)$ and Y_0, Y_1, Y_2 on $\mathbb{P}^2(K)$. The 2-uple map is $\rho_2 \colon \mathbb{P}^1(K) \to \mathbb{P}^2(K)$ given by $\rho_2(x_0 : x_1) = (x_0^2 : x_0x_1 : x_1^2)$. Using (b) its image is thus the variety $V(Y_1^2 - Y_0Y_2)$, which is the standard parabola.

For $(n,d) = (1,3)$ we have $N = 3$. Write the 3-uple map in the form $\rho_3(x_0 : x_1) = (x_0^3 : x_0^2x_1 : x_0x_1^2 : x_1^3)$. With the coordinates $(Y_0 : Y_1 : Y_2 : Y_3)$ on $\mathbb{P}^3(K)$, the image is the twisted cubic determined by the three equations $Y_0Y_2 = Y_1^2$ and $Y_0Y_3 = Y_1Y_2$ and $Y_1Y_3 = Y_2^2$.

*5. (*Discriminant locus*) To each point $a = (a_0 : \ldots : a_n) \in \mathbb{P}^n(\mathbb{C})$ associate the non-zero homogeneous polynomial

$$f_a \ := \ a_0S^n + a_1S^{n-1}T + \ldots + a_nT^n \ \in \ \mathbb{C}[S,T],$$

which is well-defined up to a factor in $\mathbb{C}^\times$. Let $D \subset \mathbb{P}^n(\mathbb{C})$ denote the set of points a for which $V(f_a)$ has cardinality less than n. Prove that D is a projective variety and find equations for it.

Solution (sketch): Let Δ denote the discriminant of the polynomial $f_a(S,1)$, which is a non-zero homogeneous polynomial in $a_0, \ldots, a_n$. By the main property of the discriminant, for any point $a = (1 : a_1 : \ldots : a_n) \in \mathbb{P}^n(\mathbb{C})$ we have $\Delta(a) = 0$ if and only if $f_a(S,1)$ has a multiple root. Thus $U_0 \cap \bar{V}(\Delta) = U_0 \cap D$. Next, a direct calculation in the case $a_0a_n \neq 0$ shows that the discriminant of the polynomial $f_a(1,T)$ is again $\pm\Delta$. Thus the same argument shows that $U_n \cap \bar{V}(\Delta) = U_n \cap D$. Finally, both the defining condition of D and the discriminant are invariant under the linear translation of coordinates $(S,T) \mapsto (S, T + \lambda S)$ for any $\lambda \in \mathbb{C}$. Use this to deduce that $\bar{V}(\Delta) = D$ everywhere.

6. Determine the closure of $V(XY - ZT) \subset K^4$ within $\mathbb{P}^4(K)$ and its singular points.

   Solution (sketch): Let x, y, z, t, u denote the coordinates on $\mathbb{P}^4(K)$ and embed K^4 into $\mathbb{P}^4(K)$ via $(x, y, z, t) \mapsto (x : y : z : t : 1)$. The closure of $V(XY - ZT)$ in $\mathbb{P}^4(K)$ is the hypersurface defined by the polynomial $f(X, Y, Z, T, U) := XY - ZT$.

   The set of singular points is $V(f, \frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}, \frac{\partial f}{\partial Z}, \frac{\partial f}{\partial T}, \frac{\partial f}{\partial U}) \in \mathbb{P}^4(K)$, which is the singleton $\{(0 : 0 : 0 : 0 : 1)\}$.