Solutions Sheet 11

Properties of Morphisms, Constructible Sets

Exercise 2 is taken from *Algebraic Geometry* by Hartshorne. Exercise 6 is adapted from *Algebraic Geometry I* by Görtz and Wedhorn.

1. Is any open subscheme of any quasicompact scheme X quasicompact? What if X is noetherian?

   Solution: Not in general. Counterexample: $X = \operatorname{Spec} k[X_1, X_2, \ldots]$ and $U = X \smallsetminus V((X_1, X_2, \ldots)) = \bigcup_{i\geqslant 1} D_{X_i}$, which is an open covering without a finite subcovering. But yes if X is noetherian: Reduce to $X = \operatorname{Spec} R$; then $U = X \smallsetminus V(\mathfrak{a})$ for a finitely generated ideal $\mathfrak{a} = (f_1, \ldots, f_n)$, and so $U = \bigcup_{i=1}^n D_{f_i}$

2. Show by example that a surjective quasi-finite morphism of finite type need not be finite.

   Solution: Counterexample Sheet 10, Problem 1.

3. Prove that any finite morphism $f\colon X \to Y$ is projective.

   Solution: It suffices to do this when Y is affine. Then X is affine, too; hence $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ for a B-algebra A that is finitely generated as a B-module. Pick generators $a_1, \ldots, a_n$, and for any $i, j = 1, \ldots, n$ write $a_i a_j = \sum_{k=1}^n b_{ijk} a_k$ with $b_{ijk} \in B$. Let $\mathfrak{a}$ be the homogeneous ideal in $B[X_0, \ldots, X_n]$ that is generated by the polynomials $f_{ij} := X_i X_j - X_0 \cdot \sum_{k=1}^n b_{ijk} X_k$ for all $i, j = 1, \ldots, n$, and consider the associated closed subscheme $X' := \bar{V}(\mathfrak{a}) \subset \mathbb{P}^n_B$. By construction its intersection with the zero-th standard open subset $X' \cap D_{X_0}$ is isomorphic to X over Y. Its intersection with the zero locus $\bar{V}(X_0)$ is $\bar{V}((X_0, \{X_i X_j \mid i, j = 1, \ldots, n\})) \subset \bar{V}((X_0, \{X_i^2 \mid i = 1, \ldots, n\}))$ and hence empty. Thus $X \cong X'$ over Y; hence X is projective over Y.

4. Let $\square$ be one of the properties quasicompact, of finite type, locally of finite type, affine, integral, finite, projective, or quasiprojective.

   (a) Show that a morphism $f\colon X \to Y$ is $\square$ if and only if there exists an open covering $Y = \bigcup_i V_i$ such that $f^{-1}(V_i) \to V_i$ is $\square$ for every i.

   *(b) Consider two morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that $g \circ f$ is $\square$. In which case does it follow that f is $\square$? Which additional condition on f or g would guarantee that?

Solution (sketch): (a) (i) *quasicompact.* Suppose f is quasicompact. Take an affine open cover $Y = \bigcup_i V_i$ such that each $f^{-1}(V_i)$ is quasicompact. Then each $f^{-1}(V_i) \to V_i$ is quasicompact because V_i is an affine open cover of itself and $f^{-1}(V_i)$ is quasicompact by assumption. Conversely, refine the open covering $Y = \bigcup_i V_i$ to an affine one $Y = \bigcup_{ij} V_{ij}$. For each i the morphism $f^{-1}(V_i) \to V_i$ is quasicompact and thus by the above, so is each $f^{-1}(V_{ij}) \to V_{ij}$. In particular, $f^{-1}(V_{ij})$ is quasicompact for all i, j since V_{ij} is open in V_{ij}; hence f is quasicompact.

(ii) *locally of finite type.* Suppose f is locally of finite type. Take any affine open cover $Y = \bigcup_i V_i$. For each i, any open affine $V \subset V_i$ is an open affine subset of Y. Thus for all open affine $U \subset f^{-1}(V)$ we have by assumption that $\mathcal{O}_{f^{-1}(V_i)}(U) = \mathcal{O}_X(U)$ is a finitely generated algebra over $\mathcal{O}_Y(V) = \mathcal{O}_{V_i}(V)$. Thus each $f^{-1}V_i \to V_i$ is locally of finite type. Conversely, refine the open covering $Y = \bigcup_i V_i$ to an affine one $Y = \bigcup_{i,j} V_{ij}$. Then, as in (i), each $f^{-1}(V_{ij}) \to V_{ij}$ is locally of finite type. In particular, for all i, j there exists an affine open cover $f^{-1}(V_{ij}) = \bigcup_{i,j,k} U_{ijk}$ such that for all i, j, k, the ring $\mathcal{O}_X(U_{ijk})$ is a finitely generated $\mathcal{O}_Y(V_{ij})$-algebra; hence f is locally of finite type.

(iii) *finite type.* By definition of 'finite type', this is just (i) and (ii).

(iv) *affine/integral/finite.* Similar to (i).

(v) *(quasi)projective.* Suppose f is (quasi)projective. Take an open covering $Y = \bigcup_i V_i$ with (locally) closed embeddings $f^{-1}(V_i) \hookrightarrow \mathbb{P}^n \times V_i$ over V_i for each i and some n. Again, since V_i is an open cover of itself, we find that $f^{-1}(V_i) \to V_i$ is (quasi)projective.

5. Let X be a scheme of finite type over a field. Show that any constructible subset of X which contains all closed points of X is equal to X.

   Solution: Since the complement of any constructible subset is constructible, it is equivalent to show that any constructible subset which contains *no* closed point of X is empty, or again that any non-empty constructible subset C contains a closed point of X. Writing C as a finite union of locally closed subsets, it suffices to show that any non-empty locally closed subscheme $Y \subset X$ contains a closed point of X. But such Y is a scheme of finite type over the given field k; so it possesses a closed point y, and the residue field $k(y)$ is a finite extension of k; hence y is also a closed point of X, as desired.

6. Let $f\colon X \to Y$ be a surjective morphism of finite type with Y noetherian. Show that a subset C of Y is constructible if and only if $f^{-1}(C)$ is constructible.

   Solution: Suppose C is constructible. Then C is a finite union of locally closed sets $C_i = U_i \cap Z_i$, where each U_i is open and each Z_i is closed. Since the underlying map of topological spaces is continuous, we find that $f^{-1}(C)$ is also of this form, i.e., constructible. Conversely, suppose $f^{-1}(C)$ is constructible. Note that $f(f^{-1}(C)) = C$ because f is surjective. Moreover f is of finite type and Y

is noetherian, and we can thus apply Chevalley's Theorem to deduce that C is constructible.

*7. Let X be of finite type over a noetherian scheme S. Show that the set of points $s \in S$ where the fiber X_s has a fixed dimension d is constructible.

Solution (sketch): The problem is local on S and X; so we may assume that $S = \operatorname{Spec} B$ for a noetherian ring B and $X = \operatorname{Spec} A$ for a finitely generated B-algebra A. We may also reduce ourselves to a reduced irreducible component of S; hence we may assume that B is an integral domain. Thereafter we may reduce ourselves to a reduced irreducible component of X; hence we may assume that A is an integral domain. By a lemma from the lecture used to prove Chevalley's theorem, there then exist an element $b \in B \smallsetminus \{0\}$, and integer $n \geqslant 0$, and an injective homomorphism $A' := B_b[T_1, \ldots, T_n] \hookrightarrow A_b$, such that A_b is a finitely generated A'-module. After replacing $X \to S$ by their open dense subschemes $\operatorname{Spec} A_b \to \operatorname{Spec} B_b$ we may assume that A is a finitely generated module over a subring $A' \cong B[T_1, \ldots, T_n]$.

For any point $s \in S$ with residue field ℓ we then have

$$\ell \hookrightarrow \ell[T_1, \ldots, T_n] \cong \underbrace{A' \otimes_B \ell}_{=:\, A'_\ell} \to \underbrace{A \otimes_B \ell}_{=:\, A_\ell},$$

and the fiber of X over s is $X_s \cong \operatorname{Spec} A_\ell$. We would like to conclude that the homomorphism $A'_\ell \to A_\ell$ is injective, but cannot do so directly, because the tensor product is only right exact in general. Nevertheless, we do know that $\operatorname{Spec} A \to \operatorname{Spec} A'$ is surjective, because $A' \subset A$ is an integral ring extension. Passing to the fibers over s it follows that $X_s = \operatorname{Spec} A_\ell \to \operatorname{Spec} A'_\ell \cong \mathbb{A}^n_\ell$ is surjective. Thus if $A'_\ell \to A_\ell$ were not injective, some non-zero polynomial $f \in A'_\ell$ would map to zero, so the corresponding morphism $X_s \to \mathbb{A}^n_\ell$ would factor through the proper closed subset $V(f) \subsetneqq \mathbb{A}^n_\ell$ and would therefore be not surjective. Therefore $A'_\ell \to A_\ell$ is injective, after all. It thus constitutes an integral ring extension, and so $\dim X_s = \dim A_\ell = \dim A'_\ell = n$. After all the preliminary reductions the desired subset of S is therefore either empty or all of S; hence it is constructible.