Solutions Sheet 13

Rational Maps, Blowups

Exercises 2 to 4 are adapted from *Algebraic Geometry* by Hartshorne.

**1. Consider schemes X and Y over a noetherian scheme S, such that Y is locally of finite type over S. Take a point $x \in X$, let $i\colon \operatorname{Spec} \mathcal{O}_{X,x} \to X$ be the canonical morphism, and consider a morphism $f\colon \operatorname{Spec} \mathcal{O}_{X,x} \to Y$ over S. Prove:

(a) There exists an open neighborhood $U \subset X$ of x and a morphism $\tilde{f}\colon U \to Y$ over S such that $\tilde{f} \circ i = f$.

(b) For any two pairs $(U, \tilde{f})$ and $(U', \tilde{f}')$ as in (a) there exists an open neighborhood $V \subset U \cap U'$ of x such that $\tilde{f}|V = \tilde{f}'|V$.

(c) Do the same conclusions hold without the assumption that S is noetherian? If not, which other condition can one put in its place?

Solution (sketch): In the lecture we proved (a) and (b) when in addition X is integral. After reduction to the case where $S = \operatorname{Spec} R$ and $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$, the point x corresponds to a prime ideal $\mathfrak{p} \subset A$ and the morphism f to a homomorphism of R-algebras $f^\flat\colon B \to A_\mathfrak{p} = \mathcal{O}_{X,x}$. Since B is a finitely generated R-algebra, the images of chosen generators b_i can all be written as $\frac{a_i}{s}$ for some $a_i \in A$ and a single $s \in A \smallsetminus \mathfrak{p}$. When X is integral, we have $A \subset A_s \subset A_\mathfrak{p}$, and so $f^\flat$ factors through a homomorphism of R-algebras $B \to A_s$, yielding the desired morphism $\tilde{f}\colon Y \to \operatorname{Spec} A_s \subset X$.

In the general case we only obtain an R-algebra homomorphism $\tilde{R} := R[X_1, \ldots, X_n] \to A_s$ sending each X_i to $\frac{a_i}{s}$, where the fraction is now taken within A_s. But as R is noetherian, so is $\tilde{R}$; hence the kernel $\mathfrak{b}$ of the R-algebra homomorphism $\tilde{R} \twoheadrightarrow B$ sending each X_i to b_i is a finitely generated ideal. Choose generators $p_1, \ldots, p_m$ of $\mathfrak{b}$ and write the image of each p_j in A_s in the form $\frac{b_j}{s^m}$ for some $b_j \in A$. Then $\frac{b_j}{s^m} \in A_s$ may be non-zero, but its image in $A_\mathfrak{p}$ is zero by assumption. By the construction of the localization $A_\mathfrak{p}$, there therefore exists a single element $t \in A \smallsetminus \mathfrak{p}$ such that $tb_j = 0$ in A for all j. Thus $\frac{b_j}{s^m}$ maps to zero in the localization A_{st}, and so we obtain an R-algebra homomorphism $\tilde{f}^\flat\colon B \cong \tilde{R}/\mathfrak{b} \to A_{st}$ whose composite with the natural homomorphism $A_{st} \to A_\mathfrak{p}$ (which still need not be injective!) is $f^\flat$. This yields the morphism $\tilde{f}\colon Y \to \operatorname{Spec} A_{st} \subset X$ desired in (a).

Since two k-algebra homomorphisms $B \to A_s$ are equal if they coincide on given generators, the fact that B is finitely generated over R suffices to prove (b) in a similar way without any noetherian hypothesis.

The proof of (a), however, seems to require *some* finiteness assumption. The same argument goes through if S is only locally noetherian, but the usual replacement is to assume that B is a *finitely presented R-algebra*, i.e., that it is isomorphic over R to $R[X_1, \ldots, X_n]/\mathfrak{b}$ for some n and some finitely generated ideal $\mathfrak{b}$. If Y possesses an open affine covering with $\operatorname{Spec} B$ of this form, the morphism $Y \to S$ is called *locally finitely presented.* See [Görtz-Wedhorn].

For a complete answer one should also decide whether finite presentation is actually necessary. I expect it is and that one can construct a counterexample by taking any finitely generated but not finitely presented R-algebra B and constructing A and $B \to A_\mathfrak{p}$ to ensure that what might go wrong in the proof does go wrong. Show me the counterexample, if you find one, or prove me wrong! (R. Pink)

2. Show that any integral scheme of finite type over a field k and of dimension r is birational to a hypersurface in $\mathbb{P}_k^{r+1}$.

   Solution: See [Hartshorne, Prop. I.4.9]

3. An integral scheme X of finite type over a field k is called *rational* if it is birationally equivalent to $\mathbb{P}_k^n$ for some n. Prove:

   (a) Any irreducible conic $C \subset \mathbb{P}_k^2$ with $C(k) \neq \varnothing$ is a rational curve.

   (b) Let X be the nodal cubic curve $V(Y^2Z - X^2(X+Z))$ in $\mathbb{P}_k^2$. Show that the projection from the point $p = (0:0:1)$ to the line $Z = 0$ induces a birational map $\varphi\colon X \dashrightarrow \mathbb{P}_k^1$. Thus X is a rational curve. What are the maximal domains of definition of φ and φ^{-1}?

   Solution (sketch): (a) The automorphism group of $\mathbb{P}_k^2$ acts transitively on sets of 3 points not on a line, so we can assume the conic C contains $(0:0:1)$, $(0:1:0)$ and $(1:0:0)$, i.e., it is of the form $V(aXY+bYZ+cXZ)$ for some $a,b,c \in k$ which are nonzero (otherwise C would be a union of two lines, contradicting the assumption that C is irreducible). Multiplying X, Y and Z by constants if necessary, we can assume $a = b = c = 1$, so C is given by $V(XY + YZ + XZ)$. In particular, this shows that all such conics are isomorphic and thus, it suffices to show that one conic is isomorphic to $\mathbb{P}^2$: for instance, the image of $\mathbb{P}^1$ under the 2-uple embedding.

   (b) The projection maps $(x:y:z) \neq (0:0:1)$ to $(x:y)$. Its inverse maps $(s:t) \neq (1:\pm 1)$ to $((t^2-s^2)s : (t^2-s^2)t : s^3)$. From this we see that φ^{-1} extends everywhere, i.e., its maximal domain of definition is $\mathbb{P}_k^1$. However, the two points $(1:\pm 1)$ are identified under φ^{-1}, which shows that φ cannot extend further; hence its maximal domain of definition is $X \smallsetminus \{(0:0:1)\}$.

4. A birational map $\varphi\colon \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^2$ over a field k is called a *plane Cremona transformation.* We give an example, called a *quadratic transformation*, given by $(a_0 : a_1 : a_2) \mapsto (a_1a_2 : a_0a_2 : a_0a_1)$ when no two of a_0, a_1, a_2 are zero.

(a) Show that φ is birational dominant and its own inverse.

(b) Find open subsets $U, V \subset \mathbb{P}^2$ such that $\varphi\colon U \to V$ is an isomorphism.

(c) Find the maximal open subset where φ is defined and the morphism on it.

Solution: Set $U := V := \{(a_0 : a_1 : a_2) \mid a_0a_1a_2 \neq 0\}$. Then φ induces the well-defined morphism $\varphi\colon U \to V$ sending $(a_0 : a_1 : a_2)$ to $(a_1a_2 : a_0a_2 : a_0a_1) \overset{!}{=} (\frac{1}{a_0} : \frac{1}{a_1} : \frac{1}{a_2})$, which evidently satisfies $\varphi^2 = \mathrm{id}_U$. Thus $\varphi\colon U \to V$ is an isomorphism, proving (b) and thereby also (a).

(c) The given formula yields a well-defined morphism $\varphi\colon W \to \mathbb{P}^2_k$ for the open subscheme $W := \mathbb{P}^2_k \smallsetminus \{(1:0:0), (0:1:0), (0:0:1)\}$. Let $L_i \subset \mathbb{P}^2_k$ denote the line with the equation $a_i = 0$. Then the formula also shows that for all points $P = (a_0 : a_1 : 0) \in L_2 \cap W$ we have $\varphi(P) = \varphi((a_0 : a_1 : 0)) = (0 : 0 : a_0a_1) = (0:0:1)$. Thus if φ extends to a neighborhood of $(1:0:0)$, by continuity the extension must map $(1:0:0)$ to the point $(0:0:1)$. But by the same argument with the last two coordinates interchanged, we have $\varphi(P) = (0:1:0)$ for all points $P \in L_1 \cap W$, and hence the extension must also map $(1:0:0)$ to the point $(0:1:0)$. This cannot hold simultaneously, so φ does not extend to a neighborhood of $(1:0:0)$. Interchanging all coordinates shows that it also does not extend to $(0:1:0)$ or $(0:0:1)$, so the maximal domain of definition is W.

*5. Let F be the surface obtained from $\mathbb{P}^2_k$ by blowing up the three points $(1:0:0)$, $(0:1:0)$, and $(0:0:1)$. Show that the plane Cremona transformation from the preceding exercise extends to an isomorphism $F \xrightarrow{\sim} F$.

(To construct F, identify each standard chart $D_{X_i} \subset \mathbb{P}^2_k$ with $\mathbb{A}^2_k$ such that the respective given point corresponds to the origin $0 \in \mathbb{A}^2_k$, construct $\tilde{D}_{X_i} \to D_{X_i}$ as in the blowup of $\mathbb{A}^2_k$ at the origin, then glue the $\tilde{D}_{X_i}$ to a scheme F over $\mathbb{P}^2_k$.)