Solutions Sheet 14

Blowups, Curves

Exercise 3 is taken from *Algebraic Geometry I* by Görtz and Wedhorn. Exercises 4 and 5 are adapted from *Algebraic Geometry* by Hartshorne.

1. Consider any coprime integers $p, q \geqslant 1$. Compute the strict transform of the affine curve $C_{p,q}\colon X^p + Y^q = 0$ under the blowup of $\mathbb{A}^2_k$ in the origin. Deduce that a finite number of iterated blowups makes this curve regular, but the number of blowups needed may be arbitrarily large.

   Solution: By the jacobian criterion, the singular points of $C_{p,q}$ are those where $X^p + Y^q = pX^{p-1} = qY^{q-1} = 0$. Since p and q are coprime, at least one of them is not divisible by the characteristic of k. Thus the last two equations imply that $X = 0$ or $Y = 0$, and then the first implies that $X = Y = 0$. Thus $C_{p,q}$ is always regular outside the origin $(0, 0)$, and it is regular there if and only if $\min\{p, q\} = 1$.

   Writing $\mathbb{A}^2_k = \operatorname{Spec} k[X, Y]$, the blowup is the union of the open charts $U := \operatorname{Spec} k[X, T]$ and $V := \operatorname{Spec} k[Y, S]$, where $Y = XT$ and $X = YS$. After substitution the given equation becomes $X^p + X^qT^q = 0$, respectively $Y^pS^p + Y^q = 0$. Without loss of generality we may assume that $p \geqslant q$. Then after clearing powers of X, respectively Y, the strict transform $\tilde{C}_{p,q}$ is defined by the equation $X^{p-q} + T^q = 0$ on U, respectively by $Y^{p-q}S^p + 1 = 0$ on V. The last equation shows that $S \neq 0$ on $\tilde{C}_{p,q} \cap V$, hence $\tilde{C}_{p,q} \subset U$. The first equation then shows that $\tilde{C}_{p,q} \cong C_{p-q,q}$.

   The passage from (p, q) to $(p - q, q)$ strictly decreases $p + q$; hence on iteration the procedure stops when $p - q = 0$. The fact that p and q are coprime then implies that $p = q = 1$. As $C_{1,1}$ is regular, the iterated blowups have made the curve regular. In the case $p \geqslant q = 1$ the number of required steps is $p - 1$, which can be arbitrarily large.

2. Determine the strict transform of the following surface when blowing up the origin. Repeat the procedure in suitable local coordinates until no singular points are left.

   (a) $V(X^2 + Y^2 + Z^3)$

   *(b) (for masochists) $V(X^2 + Y^3 + Z^5)$

   Solution: (a) By the jacobian criterion, the surface $S := V(X^2 + Y^2 + Z^3) \subset \mathbb{A}^3_k$ is regular outside the origin $(0, 0, 0)$ in characteristic $\neq 2$. If $\operatorname{char} k = 2$, then the singular locus is $V(X + Y, Z)$.

Writing $\mathbb{A}^3_k = \operatorname{Spec} k[X, Y, Z]$, consider first the open chart $W_1 := \operatorname{Spec} k[X, U, V]$ of the blowup of the origin, with $Y = XU$ and $Z = XV$. Substituting the equation $X^2 + Y^2 + Z^3 = 0$, we obtain $X^2(1 + U^2 + XV^3) = 0$. After clearing powers of X, the strict transform $\tilde{S}$ of S is defined by the equation $1 + U^2 + XV^3 = 0$ on W_1. Similarly, in the open chart $W_2 := \operatorname{Spec} k[Y, T, V]$ with $X = YT$ and $Z = YV$ we find that the strict transform is defined by the equation $T^2 + 1 + YV^3 = 0$. In the last open chart $W_3 := \operatorname{Spec} k[Z, T, U]$ with $X = ZT$ and $Y = ZU$, we obtain the equation $T^2 + U^2 + Z = 0$.

In characteristic $\neq 2$, the strict transform is regular. This can be checked in the charts using the jacobian criterion. In characteristic 2, it is regular on W_1. On W_2 and W_3 the singular locus of $\tilde{S}$ is given by $V(T + 1, V)$ and $V(U + 1, V)$, respectively.

(b) Search for the resolution of the E_8-singularity.

3. Let C be an integral curve over a field k. Show that C is proper over k if and only if its normalization $\tilde{C}$ is proper over k.

   Solution: If C is proper over k, it is in particular of finite type over k. Thus by Noether's theorem the canonical morphism $\pi \colon \tilde{C} \to C$ is finite. It is therefore proper, and so the composite morphism $\tilde{C} \to C \to \operatorname{Spec} k$ is proper, as desired.

   For the converse recall that, by the construction of the normalization, for any non-empty open affine $U = \operatorname{Spec} A \subset C$ we have $\pi^{-1}(U) = \operatorname{Spec} \tilde{A}$, where $\tilde{A}$ is the normalization of A in the function field $K(C) = \operatorname{Quot}(A)$. If $\tilde{C}$ is proper over k, it is in particular of finite type over k, so $\tilde{A}$ is a finitely generated k-algebra.

   Suppose it is generated by $a_1, \ldots, a_n$. Then each a_i is a zero of some monic polynomial $f_i[T] \in A[T]$. Let $A' \subset A$ be the k-subalgebra generated by the coefficients of $f_1, \ldots, f_n$, which by construction is a finitely generated k-algebra. Then $\tilde{A}$ is a finitely generated integral A'-algebra; hence it is finitely generated as an A'-module. Since A' is noetherian, the A'-submodule $A \subset \tilde{A}$ is then also finitely generated. Together this implies that A is a finitely generated k-algebra.

   Varying U shows that C is locally of finite type over k. Moreover, as $\tilde{C}$ is of finite type over k, it is quasi-compact, so it is the union of finitely many such $\pi^{-1}(U)$. Thus C is the union of finitely many such U; hence C is of finite type over k.

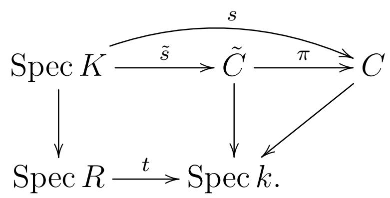
   Now we use the valuative criterion for properness. Consider any valuation ring R with field of fractions K and a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{s} & C \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \xrightarrow{t} & \operatorname{Spec} k. \end{array}$$

   If the image of t is a closed point $P \in C$, the residue field $k(P)$ is a finite extension of k. The fact that R is normal then implies that the homomorphism $s^\flat \colon k(P) \to K$

factors through R; and the corresponding morphism $\operatorname{Spec} R \to \operatorname{Spec} k(P) \to C$ is the unique lift of t making everything commute.

Assume now that the image of t is the generic point of C. Since $\pi\colon \tilde{C} \to C$ is birational, this means that we have a commutative diagram



As $\tilde{C}$ is proper over k, there exists a lift $\tilde{t}'\colon \operatorname{Spec} R \to \tilde{C}$ making everything commute; hence $\pi \circ \tilde{t}'\colon \operatorname{Spec} R \to C$ makes the original diagram commute.

Conversely, consider any morphism $\tilde{t}\colon \operatorname{Spec} R \to C$ making the original diagram commute. Then by the universal property of the normalization from Exercise Sheet 6, Problem 4, the morphism $\tilde{t}$ lifts to a unique morphism $\tilde{t}'\colon \operatorname{Spec} R \to \tilde{C}$ with $\pi \circ \tilde{t}' = \tilde{t}$. By the valuative criterion of properness for $\tilde{C}$ over k, the morphism $\tilde{t}'$ is unique; hence $\tilde{t}$ is unique as well.

By the valuative criterion it now follows that C is proper over k, as desired.

4. Let k be an algebraically closed field. Let C be a regular integral curve that is separated of finite type over k which is birational to, but not isomorphic to $\mathbb{P}^1_k$.

   (a) Show that C is isomorphic to an open subset of $\mathbb{A}^1_k$.

   (b) Show that C is affine.

   (c) Show that the affine coordinate ring $\mathcal{O}_C(C)$ is a unique factorization domain.

Solution: (a) As C is regular of dimension 1 and $\mathbb{P}^1_k$ is proper over k, any birational map $C \dashrightarrow \mathbb{P}^1_k$ over k extends uniquely to a morphism $f\colon C \to \mathbb{P}^1_k$ over k.

For any closed point $P \in C$, the residue field is a finite extension of k; hence so is the residue field of the image $f(P)$, so this image is a closed point of $\mathbb{P}^1_k$. Since C and $\mathbb{P}^1_k$ are both regular of dimension 1, the local rings at both points are discrete valuation rings. Also, since f is birational, the homomorphism of stalks $f^\flat\colon \mathcal{O}_{\mathbb{P}^1_k, f(P)} \to \mathcal{O}_{C,P}$ is injective and induces an isomorphism of quotient fields. With discrete valuation rings this is possible only when the homomorphism is an isomorphism.

Next, the valuative criterion for separatedness implies that P is uniquely determined by the subring $\mathcal{O}_{C,P} \subset K(C)$. The above isomorphism thus shows that P is determined by $f(P)$, and so f is injective.

Since f is birational, its image contains an open dense subset of $\mathbb{P}^1_k$. The complement thereof is a finite set of closed points, any subset of which is again closed;

hence the complement of the image is closed, and the image itself is open. From the explicit description of the Zariski topologies on C and $\mathbb{P}^1_k$, namely as the cofinite topology on closed points and with one additional generic point, one deduces that f is a homeomorphism from C to its image. Since f also induces isomorphisms on all stalks, it is therefore an isomorphism of schemes from C to an open subscheme of $\mathbb{P}^1_k$, in other words an open embedding.

By assumption it is not an isomorphism; hence its image misses some closed point P. Since k is algebraically closed, this point is defined over k, so after applying a suitable automorphism of $\mathbb{P}^1_k$ we may without loss of generality assume that $P = \infty$. Then we have an open embedding $C \hookrightarrow \mathbb{A}^1_k$, as desired.

(b) The complement of C is a closed set in $\mathbb{A}^1_k$, thus given by the zero set of some polynomial $f \in k[T]$, and so $C = D_f = \operatorname{Spec} k[T, f^{-1}]$ is affine.

(c) This follows from the fact that the localization $k[T, f^{-1}]$ of the unique factorization domain $k[T]$ is a unique factorization domain.

*5. Let k be an algebraically closed field of characteristic $\neq 2$. Let C be the curve $V(Y^2 - X^3 + X) \subset \mathbb{A}^2_k$. In this exercise we will show that C is not birational to $\mathbb{P}^1_k$ over k, hence its function field $K(C)$ is not a pure transcendental extension of k.

(a) Show that C is nonsingular, and deduce that its coordinate ring $A := k[X, Y]/(Y^2 - X^3 + X)$ is an integrally closed domain.

(b) Let $k[x]$ be the subring of $K := K(C)$ generated by the image of X in A. Show that $k[x]$ is a polynomial ring, and that A is the integral closure of $k[x]$ in K.

(c) Show that there is an automorphism $\sigma\colon A \to A$ which sends y to $-y$ and leaves x fixed. For any $a \in A$, define the *norm* of a to be $N(a) := a \cdot \sigma(a)$. Show that $N(a) \in k[x]$ and $N(1) = 1$ and $N(ab) = N(a) \cdot N(b)$ for any $a, b \in A$.

(d) Using the norm, show that the units in A are precisely the nonzero elements of k. Show that x and y are irreducible elements of A. Deduce that A is not a unique factorization domain.

(e) Use the previous exercise to prove that C is not birational to $\mathbb{P}^1_k$ over k.

Solution (sketch): (a) Since $Y^2 - X^3 + X$ is an irreducible polynomial in the unique factorization domain $k[X, Y]$, the coordinate ring A is an integral domain. Also C is regular by the jacobian criterion. Thus each point $p \in C$ is regular, and so each local ring $\mathcal{O}_{C,p}$ is regular and thus normal. It follows that $A = \bigcap_{p \in C} \mathcal{O}_{C,p}$ is normal, as desired.

(b) To show that $k[x]$ is a polynomial ring, check that x is transcendental using basic algebra. Since $y^2 \in k[x]$, it follows that y and thus A is contained in the integral closure of $k[x]$, and equality follows from the fact that A is normal as shown in (a).

(c) The ring homomorphism $k[X,Y] \to k[X,Y]$ defined by $X \mapsto X$ and $Y \mapsto -Y$ is an automorphism and maps the ideal $(Y^2 - X^3 + X)$ to itself. Thus it induces the desired automorphism σ.

Any $a \in A$ can be written as $a = yf + g$ for some $f, g \in k[x]$, thus $N(a) = (x - x^3)f^2 + g^2$ is in $k[x]$. The other equalities follow from the fact that σ is an automorphism of k-algebras.

(d) Clearly $k^\times \subset A^\times$. Conversely, if $a \in A$ is a unit, then $N(a)$ is a unit in $k[x]$, i.e., $N(a) \in k^\times$. Write $a = yf + g$ as in (c), so that $N(a) = (x - x^3)f^2 + g^2$. Thus

$$\deg(N(a)) = \begin{cases} 3 + 2\deg(f) & \text{if } 2 + \deg(f) \geqslant \deg(g), \\ 2\deg(g) & \text{if } 1 + \deg(f) \leqslant \deg(g). \end{cases}$$

But $N(a) \in k^\times$ implies that $\deg(N(a)) = 0$, which leaves only the second case with $f = 0$ and $\deg(g) = 0$, in other words with $a = g \in k^\times$.

The above formula also implies that there are no elements $a \in A$ with $\deg(N(a)) = 1$. Since $N(x) = x^2$ and $N(y) = x - x^3 = x(x-1)(x+1)$ are of degree 2 and 3, it follows that x and y are irreducible. Finally, since $y \cdot y = x - x^3 = x(x-1)(x+1)$ and y is not associated to x, we conclude that A cannot be a unique factorization domain.

(e) Since C is regular by (a) and not isomorphic to $\mathbb{P}^1_k$, if C were rational, then by the previous exercise A would be a unique factorization domain, which by (d) it is not.