Solutions Sheet 3

Equivalence of Categories, Representable and Adjoint Functors

1. Give an example of a functor that does not preserve monomorphisms. Dito for epimorphisms.

   Solution: Let $f : X \to Y$ be an arbitrary morphism in a category $\mathcal{C}$. Let $\mathbf{2}$ be the category with two distinct objects ξ and η and precisely one morphism $\varphi : \xi \to \eta$ besides the two identity morphisms id_ξ and id_η, and the only possible way of defining composites. Then φ is both a monomorphism and an epimorphism (but no isomorphism). Let $F : \mathbf{2} \to \mathcal{C}$ be the functor with $F\xi = X$ and $F\eta = Y$ and $F\mathrm{id}_\xi = \mathrm{id}_X$ and $F\mathrm{id}_\eta = \mathrm{id}_Y$ and $F\varphi = f$. Thus if f is neither a monomorphism nor an epimorphism, as for example for the zero morphism $\mathbb{Z} \to \mathbb{Z}$ in the category of $\mathbb{Z}$-modules, we get a functor that preserves neither monomorphisms nor epimorphisms.

   (Explanation: The property of being a mono- or epimorphism involves a quantification over all objects of a category. So if a functor is not essentially surjective on objects or not surjective on morphisms, such properties in the source category are usually not enough to guarantee the corresponding property in the target category. Note that this objection does not apply to isomorphisms, because they are defined by the explicit formulas for a two-sided inverse, not by a universal property.)

2. Consider a functor $F\colon \mathcal{C} \to \mathcal{D}$. The functor is called *fully faithful* if for all $X, X' \in \mathrm{Ob}(\mathcal{C})$ the map $F_{X,X'}\colon \mathrm{Mor}_\mathcal{C}(X, X') \to \mathrm{Mor}_\mathcal{D}(FX, FX')$ is bijective. It is called *essentially surjective* if for any $Y \in \mathrm{Ob}(\mathcal{D})$ there exists $X \in \mathrm{Ob}(\mathcal{C})$ such that $FX \cong Y$. Prove that F is part of an equivalence of categories if and only if it is fully faithful and essentially surjective.

   Solution: See Theorem IV.4.1 in *Categories for the Working Mathematician* by Saunders Mac Lane

3. Show that equivalence of categories is an equivalence relation.

   Solution (sketch): Symmetry is immediate from the definition of equivalence of categories. For reflexivity take $F = G = \mathrm{id}_\mathcal{C}$. It remains to show transitivity. Suppose $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{C}$ and $\mathcal{D} \xrightarrow{F'} \mathcal{E} \xrightarrow{G'} \mathcal{D}$ are equivalences of categories. Let $\eta\colon \mathrm{id}_\mathcal{C} \xrightarrow{\sim} GF$ and $\xi\colon \mathrm{id}_\mathcal{D} \xrightarrow{\sim} FG$, as well as $\eta'\colon \mathrm{id}_\mathcal{D} \xrightarrow{\sim} G'F'$ and $\xi'\colon \mathrm{id}_\mathcal{E} \xrightarrow{\sim} F'G'$ denote the respective natural isomorphisms of functors. Since any functor preserves isomorphisms, we have that $G\eta'_{FX}\colon GFX \to GG'F'FX$ is an isomorphism for every $X \in \mathrm{Ob}(\mathcal{C})$. Thus, so is $\theta_X := G\eta'_{FX} \circ \eta_X\colon X \to GG'F'FX$. Similarly $\theta'_X := F'\xi_{G'X} \circ \xi_X\colon X \to F'FGG'X$ is an isomorphism for each $X \in \mathrm{Ob}(\mathcal{E})$.

Check that this yields two natural isomorphisms of functors $\theta\colon \mathrm{id}_{\mathcal{C}} \xrightarrow{\sim} GG'F'F$ and $\theta'\colon \mathrm{id}_{\mathcal{E}} \xrightarrow{\sim} F'FGG'$, hence an equivalence of categories $\mathcal{C} \overset{F'F}{\to} \mathcal{E} \overset{GG'}{\to} \mathcal{C}$.

4. Show that the category of sets is not equivalent to its opposite category.

   Hint: Play around with initial and final objects and products and coproducts.

   Solution: Any equivalence with its opposite category interchanges initial with final objects and products with coproducts, and any theorem involving these translates into a dual one. It therefore suffices to find a property involving these whose corresponding dual property does not hold.

   In the category of sets the unique initial object is the empty set; the final objects are precisely the singletons $\{a\}$. The product of two objects X, Y is the cartesian product $X \times Y$, the coproduct the "exterior disjoint union" $X \times \{1\} \cup Y \times \{2\}$. The product of twice an initial object is therefore again an initial object. But the coproduct of twice a final object is a set with two elements and not a final object.

*5. Let **Cat** denote the category of small categories with functors as morphisms. Let $F\colon \mathbf{Cat} \to \mathbf{Sets}$ be the functor that sends a small category $\mathcal{C}$ to the set $\mathrm{Mor}(\mathcal{C})$. Prove that F is corepresentable.

   Solution (sketch): F is corepresented by the two-object category **2**, defined in the solution to Problem 1 above.

6. Show that the tensor product $M \otimes_R N$ corepresents the functor R-**Mod** $\to$ **Sets**, $L \mapsto \mathrm{Bilin}_R(M \times N, L)$.

   Solution: The universal property of the tensor product says that for every L the map

$$\alpha\colon \mathrm{Hom}_R(M \otimes_R N, L) \longrightarrow \mathrm{Bilin}_R(M \times N, L),\ f \mapsto (\alpha(f) : (m,n) \mapsto f(m \otimes n))$$

   is bijective. In addition, for any R-module homomorphism $g\colon L \to L'$ and any $f \in \mathrm{Hom}_R(M \otimes_R N, L)$ the calculation

$$(g \circ \alpha(f))(m,n) = g(\alpha(f)(m,n)) = g(f(m \otimes n)) = (g \circ f)(m \otimes n) = \alpha(g \circ f)(m,n)$$

   shows that the bijection is functorial in L, in other words a natural transformation of functors of L. It is thus an isomorphism of functors $\mathrm{Hom}_R(M \otimes_R N, -) \xrightarrow{\sim} \mathrm{Bilin}_R(M \times N, -)$, as desired.

*7. Fix topological spaces X and Y and consider the functor $\mathbf{Top}^{\mathrm{opp}} \to \mathbf{Sets}$, $Z \mapsto \mathrm{Mor}_{\mathbf{Top}}(Z \times Y, X)$. Under what conditions is this functor representable?

   Solution (sketch): First, consider the problem on **Sets**, i.e. for the corresponding functor $F\colon \mathbf{Sets} \to \mathbf{Sets}$, $Z \mapsto \mathrm{Mor}_{\mathbf{Sets}}(Z \times Y, X)$. Let X^Y denote the set of maps $Y \to X$. Every map $f\colon Z \times Y \to X$ corresponds to a map $Z \to X^Y$,

$z \mapsto f_z := f(z, -)$. Conversely, every map $g\colon Z \to X^Y$ induces a map $Z \times Y \to X$ by setting $g(z, y) := g(z)(y)$. The calculation

$$(g \circ (f \times \mathrm{id}_Y))(z, y) = g(f(z), y) = g(f(z))(y) = (g \circ f)(z)(y)$$

shows that this bijection $\mathrm{Mor}_{\mathbf{Sets}}(Z \times Y, X) \longleftrightarrow \mathrm{Mor}_{\mathbf{Sets}}(Z, X^Y)$ is functorial in Z. By uniqueness of representatives, we can thus deduce that X^Y represents F. For the functor in the exercise, it is natural to replace X^Y by the set $C(Y, X)$ of *continuous* functions $Y \to X$, endowed with some topology. However, neither of the maps $Z \to C(Y, X)$, $z \mapsto f_z$ and $Z \times Y \to X$, $(z, y) \mapsto g(z)(y)$ induced by f and g, respectively, are continuous in general. But a standard theorem from topology states that if $C(Y, X)$ is endowed with the compact-open topology and Y is locally compact Hausdorff, then for any topological space Z any map $Z \times Y \to X$ is continuous if and only if the induced map $Z \to C(Y, X)$ is (see, for example, Theorem 46.11 in *Topology, Second Edition* by James R. Munkres).

8. Consider the forgetful functor **Top** $\to$ **Sets** mapping any topological space to the underlying set and any continuous map to the underlying map. Does this functor have a left or right adjoint, and if so, which?

   Solution (sketch): Both: It has a left adjoint which equips a given set with the discrete topology and a right adjoint which equips a given set with the indiscrete topology.