D-MATH
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Algebraic Geometry

FS 2017

Solutions Sheet 4

Presheaves and Sheaves

Exercises 5 and 6(a) are taken or adapted from the book *Algebraic Geometry* by Hartshorne.

1. Let $X \subset \mathbb{R}^n$ be a differentiable submanifold. Let $\mathcal{F}$ be the sheaf of *normal vector fields* on X, i.e., of C^∞-functions $X \to \mathbb{R}^n$ whose values at each $x \in X$ are orthogonal to the tangent space $T_{X,x}$.

   (a) Prove that the sheaf $\mathcal{F}$ of normal vector fields on $S^{n-1} \subset \mathbb{R}^n$ is isomorphic to the sheaf of functions $C^\infty(-, \mathbb{R})$.

   (b) Give an example of a differentiable submanifold of codimension 1 where this does not hold.

   Solution (sketch): (a) For an open subset $U \subset S^{n-1}$ we define $\mathcal{F}(U) \to C^\infty(U, \mathbb{R})$ by sending f to the map $c_f \colon U \to \mathbb{R}$, $x \mapsto (f(x), x)$, where $(\cdot, \cdot)$ denotes the inner product on $\mathbb{R}^n$. This is an isomorphism with inverse map $g \mapsto (x \mapsto x \cdot g(x))$, and yields the desired isomorphism of sheaves.

   (b) An open Möbius strip in $\mathbb{R}^3$.

*2. Let X be a topological space and $j \colon U \hookrightarrow X$ the embedding of an open subset.

   (a) Prove that the functor j^{-1} on sheaves of sets or abelian groups is simply the restriction (up to isomorphy).

   (b) Show that this functor possesses a left adjoint $j_!$ (pronounced 'j shriek').

   (c) Determine the stalks of $j_!\mathcal{F}$ at all $x \in X$.

   Solution (sketch): (a) The restriction of a sheaf to open subsets of U is already a sheaf.

   (b) The sheaf $j_!\mathcal{F}$ is the sheafification of the presheaf

$$V \mapsto \begin{cases} \mathcal{F}(V) & \text{if } V \subset U, \\ \varnothing \text{ resp. } \{0\} & \text{if } V \not\subset U. \end{cases}$$

   It is called the *extension by the empty set* for sheaves of sets, and *by the zero group* for sheaves of abelian groups. This construction is functorial in $\mathcal{F}$.

   (c) The stalks of $j_!\mathcal{F}$ are those of $\mathcal{F}$ at all points of U, and the empty set or zero, respectively, on $X \setminus U$.

*3. A sheaf $\mathcal{F}$ on a topological space X is called *locally constant* if every point possesses a neighborhood U such that $\mathcal{F}|_U$ is isomorphic to a constant sheaf. Describe all locally constant sheaves of abelian groups on the circle S^1.

   Solution (sketch): The category is equivalent to the category of all pairs consisting of an abelian group M and an automorphism $\varphi\colon M \to M$, or again to the category of modules over the ring $\mathbb{Z}[X, X^{-1}]$.

4. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of sets on a topological space X.

   (a) Show that f is a monomorphism in the category of sheaves if and only if f is injective on the level of stalks.

   (b) Show that f is an epimorphism in the category of sheaves if and only if f is surjective on the level of stalks.

   (c) Give an example of an epimorphism of sheaves f and an open set $U \subset X$ such that f_U is not surjective.

   Solution (sketch): (a) For both directions, use the proposition from the lecture stating that f is injective on the level of stalks if and only if f is injective on the level of open sets. For the 'only if' part, use an 'indicator sheaf' of a subset U with one section over every open set contained in U, and no section over any other open set.

   (b) For the 'only if' part, use a skyscraper sheaf with value $\{a, b, c\}$ at some fixed point $x \in X$.

   (c) Let $X := \mathbb{C} \smallsetminus \{0\}$, and consider $\mathcal{F}$ the sheaf of meromorphic functions and $\mathcal{G}$ the sheaf of nowhere-zero meromorphic functions. Let $f\colon \mathcal{F} \to \mathcal{G}$ be post-composition with $\exp\colon \mathbb{C} \to \mathbb{C}$; then f is an epimorphism of sheaves but fails to be surjective on (global) sections: for instance, there is no holomorphic function $\varphi\colon X \to \mathbb{C}$ such that $\exp \varphi(z) = z$ for all non-zero z.

5. (a) Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups on a topological space X.

      (i) Show that the presheaf $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ is a sheaf. It is called the *direct sum* of $\mathcal{F}$ and $\mathcal{G}$, and is denoted by $\mathcal{F} \oplus \mathcal{G}$.

      (ii) Show that it plays the role of direct sum and direct product in the category of sheaves of abelian groups on X.

   (b) Let $\{\mathcal{F}_i\}_i$ be an inverse system of sheaves of abelian groups on X.

      (i) Show that the presheaf $U \mapsto \varprojlim_i \mathcal{F}_i(U)$ is already a sheaf. It is called the *inverse limit* of the system $\{\mathcal{F}_i\}_i$, and is denoted by $\varprojlim_i \mathcal{F}_i$.

      (ii) Show that it has the universal property of an inverse limit in the category of sheaves.

Solution (sketch): (a) Take a cover $\{U_i\}_{i\in I}$ of an open set $U \subset X$. First, we check uniqueness. Let $s, t \in \mathcal{F}(U) \oplus \mathcal{G}(U)$ with $s|_{U_i} = t|_{U_i}$ for all $i \in I$. Since s and t are elements of a direct sum of abelian groups, we have unique decompositions $s = s_{\mathcal{F}} + s_{\mathcal{G}}$ and $t = t_{\mathcal{F}} + t_{\mathcal{G}}$ with $s_{\mathcal{F}}, t_{\mathcal{F}} \in \mathcal{F}(U)$ and $s_{\mathcal{G}}, t_{\mathcal{G}} \in \mathcal{G}(U)$, and the equalities $s_{\mathcal{F}}|_{U_i} + s_{\mathcal{G}}|_{U_i} = s|_{U_i} = t|_{U_i} = t_{\mathcal{F}}|_{U_i} + t_{\mathcal{G}}|_{U_i}$ imply $s_{\mathcal{F}}|_{U_i} = t_{\mathcal{F}}|_{U_i}$ and $s_{\mathcal{G}}|_{U_i} = t_{\mathcal{G}}|_{U_i}$ for each $i \in I$. Since $\mathcal{F}|_U$ and $\mathcal{G}|_U$ are sheaves, it follows that $s_{\mathcal{F}} = t_{\mathcal{F}}$ and $s_{\mathcal{G}} = t_{\mathcal{G}}$ on U; hence $s = t$ on U. For existence, let $\{s_i \in \mathcal{F}(U) \oplus \mathcal{G}(U)\}_{i\in I}$ be a compatible collection of sections. As above, decompose each $s_i = s_{i,\mathcal{F}} + s_{i,\mathcal{G}}$ and find the unique $s_{\mathcal{F}}$ and $s_{\mathcal{G}}$ that exist because $\mathcal{F}|_U$ and $\mathcal{G}|_U$ are sheaves. Then $s := s_{\mathcal{F}} + s_{\mathcal{G}}$ does the trick and it follows that $\mathcal{F} \oplus \mathcal{G}$ is indeed a sheaf.

(b) Again, take a cover $\{U_j\}_{j\in J}$ of an open set $U \subset X$ and for all $j \in J$ let $s_j = (s_j^i)_i \in \varprojlim_i \mathcal{F}_i(U_j)$ be a collection of sections compatible on intersections. Since each $\mathcal{F}_i$ is a sheaf, the s_j^i glue to a section $s^i \in \mathcal{F}_i(U)$ for each i. Let $\varphi_{ik}\colon \mathcal{F}_k \to \mathcal{F}_i$ be a transition map of the inverse system. We have $\varphi_{ik,U_j}(s_j^k) = (s_j^i)$ for all $j \in J$ and thus for all $x \in U_j$ we obtain $\varphi_{ik}(s_k)_x = (\varphi_{ik})_x(s_{k,x}) = s_{i,x}$. It follows that the s^i are compatible with the transition maps, hence $s := (s^i)_i$ is an element of $\varprojlim_i \mathcal{F}_i(U)$ restricting to s_j over each U_j.

For the universal property, let $\{f_i\colon \mathcal{G} \to \mathcal{F}_i\}_i$ be a collection of morphism compatible with the inverse system. Define $f\colon \mathcal{G} \to \varprojlim_i \mathcal{F}_i$ by $f_U\colon \mathcal{G}(U) \to \varprojlim_i \mathcal{F}_i(U)$, $s \mapsto (f_{i,U}(s))_i$. Check that this is a welldefined morphism and verify uniqueness.

6. (a) Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups on X. For any open set $U \subset X$, show that the set $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ of morphisms of the restricted sheaves has a natural structure of an abelian group. Show that the presheaf $\mathscr{H}om(\mathcal{F}, \mathcal{G})$ given by $\mathscr{H}om(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf.

   (b) Show that the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathbb{Z}} \mathcal{G}(U)$ does not define a sheaf in general. (Its sheafification is called the *tensor product sheaf* $\mathcal{F} \otimes_{\mathbb{Z}} \mathcal{G}$.)

Solution (sketch): (a) For morphisms of sheaves $f, g \in \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ define $f + g\colon \mathcal{F}|_U \to \mathcal{G}|_U$ by $(f + g)_V := f_V + g_V$ for all open $V \subset U$. Since f_V and g_V are group homomorphisms, this is welldefined. This defines an element of $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ because the restriction maps are homomorphisms of abelian groups. To check that $\mathscr{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf, let $U \subset X$ be an open set and let $U = \bigcup_{i\in I} U_i$ be an open covering. Let $\{f_i \in \mathrm{Hom}(\mathcal{F}|_{U_i}\mathcal{G}|_{U_i})\}_{i\in I}$ be a collection of morphisms compatible on intersections. Define $f\colon \mathcal{F}|_U \to \mathcal{G}|_U$ as follows: for any $V \subset U$ consider the cover $\{V_i := V \cap U_i\}_{i\in I}$. For $s \in \mathcal{F}(V)$ write $s_i := s|_{V_i}$ and $t_i := f_{i,V_i}(s_i)$ for each $i \in I$. Since the f_i are compatible on overlaps, so are the t_i, which thus glue to $t \in \mathcal{G}(V)$. Set $f_V(s) := t$. This is compatible with the restriction maps: Indeed, if $V' \subset V$, then $f_V(s|_{V_i})$ is the gluing of $f_{i,V'\cap V_i}(s_i|_{V'\cap V_i}) = t_i|_{V'\cap V_i}$ because the f_i are compatible with the restriction maps. The desired comptibility now follows from the equality $(t|_{V'})|_{V_i} = t_i$.

(b) Take the example of 1(b), but restrict to normal vectors of integral lengths, obtaining a sheaf of $\mathbb{Z}$-modules $\mathcal{G}$, all of whose stalks are non-canonically isomorphic to $\mathbb{Z}$. Show that $\mathcal{G} \otimes_{\mathbb{Z}} \mathcal{G}$ is isomorphic to the constant sheaf $\underline{\mathbb{Z}}$. Thus $(\mathcal{G} \otimes_{\mathbb{Z}} \mathcal{G})(X) \cong \mathbb{Z}$, whereas $\mathcal{G}(X) = 0$ and hence $\mathcal{G}(X) \otimes_{\mathbb{Z}} \mathcal{G}(X) = 0$.

**7. Discuss: Under which conditions does the sheafification of a presheaf in an arbitrary category exist?

Solution: See for example: `https://ncatlab.org/nlab/show/sheafification`