Solutions Sheet 5

Sheaves of Abelian Groups, Locally Ringed Spaces

Exercises 1, 4 and 5(c) are taken or adapted from *Algebraic Geometry* by Hartshorne. Exercises 3(a,b) and 5 are from *Algebraic Geometry I* by Görtz and Wedhorn.

1. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups. Show that $\operatorname{im}(f) \cong \mathcal{F}/\ker(f)$ and $\operatorname{coker}(f) \cong \mathcal{G}/\operatorname{im}(f)$.

   Solution: By definition, the *quotient sheaf* of a sheaf of abelian groups $\mathcal{F}$ by a sheaf of subgroups $\mathcal{F}'$ is the cokernel of the inclusion homomorphism of sheaves $\mathcal{F}' \hookrightarrow \mathcal{F}$, that is, the sheafification of the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{F}'(U)$.

   Also by definition, the *image sheaf* $\operatorname{im}(f)$ is the sheafification of the presheaf $U \mapsto \operatorname{im}(f_U\colon \mathcal{F}(U) \to \mathcal{G}(U))$, which is isomorphic to the presheaf $U \mapsto \mathcal{F}(U)/\ker(f_U)$. As the kernel presheaf $\ker(f)\colon U \mapsto \ker(f_U)$ is already a sheaf, it follows that $\operatorname{im}(f)$ is isomorphic to the quotient sheaf $\mathcal{F}/\ker(f)$.

   For the cokernel the situation is more complicated, because the image presheaf is not necessarily a sheaf. So we first note that sheafification is an exact functor, namely: Consider any sequence $\mathcal{F} \to \mathcal{F}' \to \mathcal{F}''$ of presheaves of abelian groups such that $\mathcal{F}(U) \to \mathcal{F}'(U) \to \mathcal{F}''(U)$ is exact for any open $U \subset X$. Then by the exactness of filtered direct limits, the associated sequence of stalks $\mathcal{F}_x \to \mathcal{F}'_x \to \mathcal{F}''_x$ is exact for every $x \in X$. By a proposition from the lecture the induced sequence of sheafifications $\tilde{\mathcal{F}} \to \tilde{\mathcal{F}}' \to \tilde{\mathcal{F}}''$ has the same stalks, and by another it is therefore an exact sequence of sheaves.

   Back in our situation, for every U we have $\operatorname{im}(f_U) \subset \operatorname{im}(f)(U)$ and hence an exact sequence

   $$0 \longrightarrow \operatorname{im}(f_U) \longrightarrow \operatorname{im}(f)(U) \longrightarrow \operatorname{coker}(f_U) \longrightarrow \mathcal{G}(U)/\operatorname{im}(f)(U) \longrightarrow 0.$$

   By sheafification this yields an exact sequence of sheaves. But as $\operatorname{im}(f)$ is already the sheafification of the presheaf $U \mapsto \operatorname{im}(f_U)$, the second morphism from the left becomes an isomorphism of sheaves, hence the middle one becomes zero and the next one an isomorphism. Therefore the *cokernel sheaf* $\operatorname{coker}(f)$, defined as the sheafification of the presheaf $U \mapsto \operatorname{coker}(f_U)$, is isomorphic to the sheafification of the presheaf $U \mapsto \mathcal{G}(U)/\operatorname{im}(f)(U)$, that is, the quotient sheaf $\mathcal{G}/\operatorname{im}(f)$, as desired.

2. Let X and Y be topological spaces and $f\colon X \to Y$ a continuous map.

   (a) Show that for any exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ in $\mathbf{Sh}_{\mathbf{Ab}}(X)$ and any open subset $U \subset X$, the sequence $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$ is exact.

(b) Let $\mathcal{F}$ be a sheaf of abelian groups on X. Show that $\mathscr{H}om(\mathcal{F}, -)$ is a left exact covariant functor and that $\mathscr{H}om(-, \mathcal{F})$ is a left exact contravariant functor $\mathbf{Sh_{Ab}}(X) \to \mathbf{Sh_{Ab}}(X)$.

(c) Show that for any exact sequence $0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ in $\mathbf{Sh_{Ab}}(X)$, the sequence $0 \longrightarrow f_*\mathcal{F} \xrightarrow{f_*\varphi} f_*\mathcal{G} \xrightarrow{f_*\psi} f_*\mathcal{H}$ is exact.

(d) Show that for any sheaf of abelian groups $\mathcal{G}$ on Y and any $x \in X$ there is a natural isomorphism of stalks $\mathcal{G}_{f(x)} \cong (f^{-1}\mathcal{G})_x$.

(e) Show that f^{-1} is an exact functor from sheaves of abelian groups on Y to sheaves of abelian groups on X.

Solution: (a) By a proposition from the lecture, we know that the sequence of sheaves is exact if and only if the sequence of stalks $0 \to \mathcal{F}_x \xrightarrow{\varphi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$ is exact for every $x \in X$. By another proposition from the lecture φ_x is injective for every $x \in X$ if and only if φ_U is for every open $U \subset X$. Thus, the sequence $0 \to \mathcal{F}(U) \xrightarrow{\varphi_U} \mathcal{G}(U) \xrightarrow{\psi_U} \mathcal{H}(U)$ is exact at $\mathcal{F}(U)$. For exactness at $\mathcal{G}(U)$, let $s \in \mathcal{F}(U)$. Since the sequence is exact on stalks, we have $(\psi_U \circ \varphi_U(s))_x = \psi_x \circ \varphi_x(s_x) = 0$ for every $x \in U$ and thus by the sheaf property $\psi_U \circ \varphi_U = 0$, i.e., $\operatorname{im} \varphi_U \subset \ker \psi_U$. For the reverse inclusion, let $t \in \ker \psi_U$. For each $x \in U$, there is a germ $s_x \in \mathcal{F}_x$ such that $\varphi_x(s_x) = t_x$, and thus an open subset $V_x \subset U$ and a section $s_{V_x} \in \mathcal{F}(V_x)$ such that $\varphi_{V_x}(s_{V_x}) = t|_{V_x}$. The subsets V_x for $x \in U$ comprise an open cover of U and on intersections $W := V_x \cap V_y$ with $x, y \in U$ we have that φ_W is injective and $\varphi_W(s_{V_x}|_W) = t|_W = \varphi_W(s_{V_y}|_W)$. Thus, by the sheaf condition, we can glue to a section $s \in \mathcal{F}(U)$ such that $s|_{V_x} = s_{V_x}$ for each $x \in U$ and $\varphi_U(s) = t$; hence $\ker \psi_U \subset \operatorname{im} \varphi_U$.

(b) We know from Exercise Sheet 4 that $\mathscr{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf of abelian groups for all sheaves of abelian groups $\mathcal{G}$. Check that $\mathscr{H}om(\mathcal{F}, -)$ satisfies the definition of a covariant functor. To show that this is left exact, let $0 \to \mathcal{G} \xrightarrow{\varphi} \mathcal{G}' \xrightarrow{\varphi'} \mathcal{G}''$ be an exact sequence of sheaves of abelian groups. We want to show that the sequence of sheaves $0 \to \mathscr{H}om(\mathcal{F}, \mathcal{G}) \xrightarrow{\varphi \circ (-)} \mathscr{H}om(\mathcal{F}, \mathcal{G}') \xrightarrow{\varphi' \circ (-)} \mathscr{H}om(\mathcal{F}, \mathcal{G}'')$ is exact, which again we can check on stalks. Now check that $\mathscr{H}om(\mathcal{F}, \mathcal{G})_x = \mathscr{H}om(\mathcal{F}_x, \mathcal{G}_x)$ and the same for $\mathcal{G}'$ and $\mathcal{G}''$ in place of $\mathcal{G}$. Thus we must show that the sequence $0 \to \mathscr{H}om(\mathcal{F}_x, \mathcal{G}_x) \xrightarrow{\varphi_x \circ (-)} \mathscr{H}om(\mathcal{F}_x, \mathcal{G}'_x) \xrightarrow{\varphi'_x \circ (-)} \mathscr{H}om(\mathcal{F}_x, \mathcal{G}''_x)$ is exact, which is a standard fact from commutative algebra.

(c) Since the sequence $0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ in $\mathbf{Sh_{Ab}}(X)$ is exact, by (a) so is $0 \longrightarrow \mathcal{F}(U) \xrightarrow{\varphi_U} \mathcal{G}(U) \xrightarrow{\psi_U} \mathcal{H}(U)$ for every open $U \subset X$. In particular, this holds for all open sets of the form $f^{-1}(V)$ for V open in Y, in other words, the sequence $0 \longrightarrow f_*\mathcal{F}(V) \xrightarrow{f_*\varphi_V} f_*\mathcal{G}(V) \xrightarrow{f_*\psi_V} f_*\mathcal{H}(V)$ is exact.

(d) Since $f^{-1}\mathcal{G}$ is the sheafification of the inverse image presheaf $f^+\mathcal{G}$, for any

$x \in X$ we have a natural isomorphism of stalks

$$(f^{-1}\mathcal{G})_x \cong (f^+\mathcal{G})_x := \varinjlim_{\substack{U\subset X \text{ open} \\ x\in U}} (f^+\mathcal{G})(U) = f^+(\varinjlim_{\substack{U\subset X \text{ open} \\ x\in U}} \mathcal{G}(U)) = f^+(\mathcal{G}_x) = \mathcal{G}_{f(x)}$$

(e) We already know from the lecture that f^{-1} is a functor $\mathbf{Sh_{Ab}}(Y) \to \mathbf{Sh_{Ab}}(X)$. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence in $\mathbf{Sh_{Ab}}(Y)$. Then for any $y \in Y$ the sequence $0 \to \mathcal{F}_y \to \mathcal{G}_y \to \mathcal{H}_y \to 0$ is exact. Use naturality of the isomorphisms in (d) to show that $0 \to (f^{-1}\mathcal{F})_x \to (f^{-1}\mathcal{G})_x \to (f^{-1}\mathcal{H})_x \to 0$ is exact for every $x \in X$. Thus the sequence of sheaves $0 \to f^{-1}\mathcal{F} \to f^{-1}\mathcal{G} \to f^{-1}\mathcal{H} \to 0$ is exact.

3. Consider $\mathbb{C}$ with the analytic topology. Denote by $\mathcal{O}_\mathbb{C}$ the sheaf of holomorphic functions on $\mathbb{C}$, and let $\mathcal{O}_\mathbb{C}^\times$ be the sheaf of nowhere vanishing holomorphic functions. Let $\underline{\mathbb{Z}}$ and $\underline{\mathbb{C}}$ denote the constant sheaves on $\mathbb{C}$ with values in $\mathbb{Z}$, and $\mathbb{C}$, respectively.

   (a) Show that $(\mathbb{C}, \mathcal{O}_\mathbb{C})$ is a locally ringed space. What are the residue fields $k(z)$ for $z \in \mathbb{C}$?

   (b) Let $D\colon \mathcal{O}_\mathbb{C} \to \mathcal{O}_\mathbb{C}$ be the morphism of sheaves which, for $U \subset \mathbb{C}$ open, sends $f \in \mathcal{O}_\mathbb{C}(U)$ to its derivative $f' \in \mathcal{O}_\mathbb{C}(U)$. Show that there is a natural exact sequence of sheaves

$$0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \mathcal{O}_\mathbb{C} \xrightarrow{D} \mathcal{O}_\mathbb{C} \longrightarrow 0.$$

   (c) Show that there is a natural exact sequence of sheaves

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_\mathbb{C} \xrightarrow{f\mapsto e^{2\pi i f}} \mathcal{O}_\mathbb{C}^\times \longrightarrow 1.$$

   (d) Which of the sequences in (b) and (c) are exact sequences in the sense of presheaves?

Solution (sketch): (a) For any $z \in \mathbb{C}$, the stalk $\mathcal{O}_{\mathbb{C},z}$ is isomorphic to the ring $\mathbb{C}\{T - z\}$ of power series with a positive radius of convergence. This is a local ring with maximal ideal $\mathfrak{m}_z$ consisting of all power series with constant coefficient $a_0 = 0$. Thus $(\mathbb{C}, \mathcal{O}_\mathbb{C})$ is a locally ringed space with residue fields $k(z) \cong \mathbb{C}\{T - z\}/\mathfrak{m}_z \cong \mathbb{C}$ for all $z \in \mathbb{C}$.

(b, c) We again check exactness on stalks, i.e., of the sequences

$$0 \longrightarrow \mathbb{C} = \underline{\mathbb{C}}_z \longrightarrow \mathcal{O}_{\mathbb{C},z} \xrightarrow{D_z} \mathcal{O}_{\mathbb{C},z} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z} = \underline{\mathbb{Z}}_z \longrightarrow \mathcal{O}_{\mathbb{C},z} \xrightarrow{f\mapsto e^{2\pi i f}} \mathcal{O}_{\mathbb{C},z}^\times \longrightarrow 1$$

for all $z \in \mathbb{C}$, which are direct calculations of power series.

(d) Both sequences are not exact on $U := \mathbb{C}\smallsetminus\{0\}$. For (b) observe that $\frac{1}{z} \in \mathcal{O}_{\mathbb{C}}(U)$, but any solution of the equation $f'(z) = \frac{1}{z}$ would have to be a branch of the logarithm function $\log z$ over U, which does not exist. Similarly, for (c) any solution of the equation $e^{2\pi i f(z)} = z$ is locally of the form $f(z) = \frac{\log z}{2\pi i} + n$ for some $n \in \mathbb{Z}$, but again no branch of the logarithm exists over all of U.

4. Let R be a ring and $X := \operatorname{Spec} R$. Show that for any $a \in R$, the locally ringed space $(D_a, \mathcal{O}_X|_{D_a})$ is isomorphic to $(\operatorname{Spec} R_a, \mathcal{O}_{\operatorname{Spec} R_a})$.

   Solution (sketch): Let $\iota\colon R \to R_a$ be the localisation homomorphism. Show that the induced map $f\colon \operatorname{Spec} R_a \to D_a$, $\mathfrak{p} \mapsto \iota^*\mathfrak{p}$ is a homeomorphism and that the induced morphism of sheaves $f^\flat\colon \mathcal{O}_X|_{D_a} \to f_*\mathcal{O}_{\operatorname{Spec} R_a}$ is an isomorphism (this can be checked on stalks).

5. Let $(X, \mathcal{O}_X)$ be a locally ringed space.

   (a) Let $U \subset X$ be an open and closed subset. Show that there exists a unique section $e_U \in \mathcal{O}_X(X)$ such that $e_U|_V = 1$ for all open subsets V of U and $e_U|_V = 0$ for all open subsets V of $X \smallsetminus U$. Show that $U \mapsto e_U$ yields a bijection between the set of open and closed subsets of X and the set of idempotent elements of the ring $\mathcal{O}_X(X)$.

   (b) Show that $e_U e_{U'} = e_{U\cap U'}$ for all open and closed subsets $U, U' \subset X$.

   (c) Suppose $X = \operatorname{Spec} R$ for some ring R. Prove that the following are equivalent:

      (i) X is non-empty and not connected.

      (ii) There exist nonzero elements $e_1, e_2 \in R$ such that $e_1e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 + e_2 = 1$ (these elements are called *orthogonal idempotents)*.

      (iii) R is isomorphic to a direct product $R_1 \times R_2$ of nonzero rings.

   (d) Let R be a local ring. Show that $\operatorname{Spec} R$ is connected.

   Solution (sketch): (a) For $U \subset X$ open and closed $X = U \cup (X \smallsetminus U)$ is an open disjoint covering of X. Thus, by the sheaf property, we can glue the sections $1 \in \mathcal{O}_X(U)$ and $0 \in \mathcal{O}_X(X \smallsetminus U)$ to a unique section $e_U \in \mathcal{O}_X(X)$ with $e_U|_U = 1$ and $e_U|_{X\smallsetminus U} = 0$, and hence with $e_U|_V = 1$ for $V \subset U$ open and $e_U|_V = 0$ for $V \subset X \smallsetminus U$ open. Moreover e_U is idempotent because $e_U^2|_U = (e_U|_U)^2 = 1$ and $e_U^2|_{X\smallsetminus U} = (e_U|_{X\smallsetminus U})^2 = 0$ and thus by uniqueness $e_U^2 = e_U \in \mathcal{O}_X(X)$. This yields a well-defined map

   $$\{\text{open and closed subsets of } X\} \to \{\text{idempotent elements of } \mathcal{O}_X(X)\}, \quad U \mapsto e_U.$$

   To define an inverse, let $e \in \mathcal{O}_X(X)$ be idempotent and consider the sets $U_e := \{x \in X \mid e_x = 1 \in \mathcal{O}_{X,x}\}$ and $V_e := \{x \in X \mid e_x = 0 \in \mathcal{O}_{X,x}\}$. These sets are both open and $U_e \cap V_e = \varnothing$. We claim that $V_e = X \smallsetminus U_e$ and U_e is thus also closed. Since $(X, \mathcal{O}_X)$ is a locally ringed space, this follows from the fact that for any local ring R, the only idempotent elements are 0 and 1. Finally, since local sections are determined by their stalks, we conclude that $U_{e_U} = U$ for any open and closed set $U \subset X$ and hence, the above map is a bijection.

(b) Check that $e_U e_{U'}|_V = 1$ for $V \subset U \cap U'$, and $e_U e_{U'}|_V = 0$ for $V \subset X \smallsetminus (U \cap U')$. By definition, this is also true for $e_{U \cap U'}$; hence $e_U e_{U'} = e_{U \cap U'}$ by uniqueness proved in (a).

(c) (i)$\Rightarrow$(ii): If X is not connected, then there exists a nonempty proper subset $U \subsetneqq X$ which is open and closed. Take the idempotent $e_U \in \mathcal{O}_X(X)$ from part (a) and check that $e_{X \smallsetminus U} = 1 - e_U \in \mathcal{O}_X(X)$. These are two orthogonal idempotents in $\mathcal{O}_X(X) = R$.

(ii)$\Rightarrow$(iii): Check that $R_1 := e_U \mathcal{O}_X(X)$ and $R_2 := (1 - e_U)\mathcal{O}_X(X)$ are nonzero rings and that the map $\mathcal{O}_X(X) = R \to R_1 \times R_2$ given by $s \mapsto (e_U s, (1 - e_U)s)$ is a ring isomorphism with inverse $(r, r') \mapsto r + r'$.

(iii)$\Rightarrow$(i): Identify $\mathcal{O}_X(X) = R$ with $R_1 \times R_2$ and consider the idempotent element $e := (1, 0) \in \mathcal{O}_X(X)$. The associated open and closed subset U_e is proper and nonempty because $e \neq 0, 1$. Thus $X = U \sqcup (X \smallsetminus U)$ is not connected.

(d) Since R is local, the only idempotents in R are 0 and 1. So (c)(ii) cannot hold, hence $\operatorname{Spec} R$ is connected.