Solutions Sheet 6

Schemes and Subschemes

Exercises 1 and 4 are taken or adapted from *Algebraic Geometry* by Hartshorne. Exercises 2, 3, 6 and 7 are from *Algebraic Geometry I* by Görtz and Wedhorn.

1. Let $\varphi\colon R \to S$ be a ring homomorphism and let $f\colon Y := \operatorname{Spec} S \to \operatorname{Spec} R =: X$ be the induced morphism of affine schemes.

   (a) Show that φ is injective if and only if the homomorphism of sheaves $f^\flat\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ is. Show furthermore in that case $f(Y)$ is dense in X.

   (b) Show that if φ is surjective, then f is a homeomorphism of Y onto a closed subset of X and $f^\flat$ is surjective.

   (c) Prove the converse to (b), namely, if $f\colon Y \to X$ is a homeomorphism onto a closed subset and $f^\flat\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective, then φ is surjective.
   Hint: Consider $X' := \operatorname{Spec}(R/\ker\varphi)$ and use (a) and (b).

   Solution: (a) The 'if' part follows from the fact that $f^\flat_X = \varphi$, using a proposition from the lecture. Conversely, if φ is injective, then so is the induced map on localizations $R_a \to S_{\varphi(a)}$ for every $a \in R$. These maps are the maps $f^\flat_{D_a}$, so f is injective on a base of open sets, hence on every stalk. Exercise 4(a), Sheet 4 now implies that f is injective. To show that $f(Y)$ is dense in X, it suffices to show that for any nonempty basic open set D_a with $a \in R$, the intersection $D_a \cap f(Y)$ is nonempty, i.e., there exists some $\mathfrak{p} \in \operatorname{Spec} S$ such that $f(\mathfrak{p}) \in D_a$. Suppose not. Using $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$ and the definition of D_a, this means that $a \in \varphi^{-1}(\mathfrak{p})$ for every prime $\mathfrak{p} \subset S$ and thus $\varphi(a) \in \bigcap_{\mathfrak{p}\in\operatorname{Spec} S} \mathfrak{p} = \operatorname{rad} S$. So $\varphi(a)$ is nilpotent and since φ is injective, it follows that a is nilpotent. By Commutative Algebra this implies that D_a is empty.

   (b) Note that $\varphi\colon R \to S$ factors as $R \overset{\pi}{\twoheadrightarrow} R/\ker\varphi \overset{\tilde\varphi}{\hookrightarrow} S$. By equivalence of categories, it follows that $(f, f^\flat)$ factors as $(g, g^\flat) \circ (h, h^\flat)$, where $(g, g^\flat)$ is the morphism corresponding to π and $(h, h^\flat)$ is the morphism corresponding to $\tilde\varphi$. If φ is surjective, then $\tilde\varphi$ is an isomorphism and thus $h\colon \operatorname{Spec} S \to \operatorname{Spec}(R/\ker\varphi) =: X'$ is a homeomorphism, again by equivalence of categories. Check that g is a homeomorphism of X' onto the closed subset $V(\ker\varphi) \subset \operatorname{Spec} R = X$ and thus f is a homeomorphism of Y onto $V(\ker\varphi) \subset X$. For surjectivity of $f^\flat$, it suffices to check this on stalks. Let $x \in X$ and $s_x \in (f_*\mathcal{O}_Y)_x$. Let D_y be a basic open set containing x and let $s \in \mathcal{O}_Y(f^{-1}(D_a))$ be a section with germ s_x at x. Note that $\mathcal{O}_Y(f^{-1}(D_a)) = \mathcal{O}_Y(D_{f(a)}) = S_{\varphi(a)}$. Since φ is surjective, so is the induced homomorphism on localizations $R_a \to S_{\varphi(a)}$, which is given by $f^\flat_{D_a}$. Thus we can choose a section $t \in \mathcal{O}_X(D_a) = R_a$ which is mapped to s under $f^\flat_{D_a}$ and we obtain $t_x = (f^\flat_{D_a}(s))_x = f^\flat_x(s_x)$; hence $f^\flat$ is surjective on stalks and therefore surjective.

(c) Consider again the morphisms $(g, g^\flat)$ and $(h, h^\flat)$ from part (b). Since $\pi\colon R \twoheadrightarrow R/\ker\varphi$ is surjective and $\tilde{\varphi}$ is injective, by parts (a) and (b) we have that $h(Y)$ is dense in X' and g is a homeomorphism of X' onto a closed subset of X. Since by assumption f is a homeomorphism of Y onto a closed subset of X, it follows that $h(Y)$ is both dense and closed in X', thus h is surjective and from $f = g \circ h$ it follows that h is a homeomorphism $Y \to X'$. Also by parts (a) and (b) we have that $h^\flat$ is injective and $g^\flat$ is surjective. Since $f^\flat$ is by assumption surjective, it follows that $h^\flat$ is a bijection. Together with h being a homeomorphism we conclude that $\operatorname{Spec} S \cong \operatorname{Spec}(R \ker\varphi)$ and thus $\tilde{\varphi}\colon R/\ker\varphi \to S$ is an isomorphism by equivalence of categories, hence φ is surjective.

2. Let $(R_i)_{i\in I}$ be a family of nonzero rings.

   (a) Prove that $\coprod_{i\in I} \operatorname{Spec} R_i \cong \operatorname{Spec}(\prod_{i\in I} R_i)$ if I is finite.

   (b) Prove that $\coprod_{i\in I} \operatorname{Spec} R_i$ is not an affine scheme if I is infinite.
   Hint: Test for quasi-compactness.

Solution (sketch): (a) By a proposition in the lecture, the functor Spec is part of a contravariant equivalence from **Rings** to the category of affine schemes. In particular, it sends products to coproducts, and vice versa. The product in **Rings** is the cartesian product and the coproduct in the category of affine schemes is disjoint union. More explicitly, reduce to the case where $|I| = 2$. Then a bijection between $\operatorname{Spec}(R_1 \times R_2)$ and $\operatorname{Spec}(R_1) \sqcup \operatorname{Spec}(R_2)$ results from the fact that the prime ideals of $R_1 \times R_2$ are precisely those of the form $\mathfrak{p} \times R_2$ where $\mathfrak{p}$ is a prime ideal of R_1, or $R_1 \times \mathfrak{q}$ where $\mathfrak{q}$ is a prime ideal of R_2.

(b) By a proposition in the lecture, any affine scheme is quasicompact. However, an infinite disjoint union of nonempty quasicompact spaces is clearly not quasicompact; hence $\coprod_{i\in I} \operatorname{Spec} R_i$ cannot be an affine scheme.

3. Let R be a principal ideal domain (or more generally a Dedekind ring), and let $f \in R$ be a nonzero element. Describe the affine scheme $X := \operatorname{Spec} R/fR$ (its underlying topological space, the stalks $\mathcal{O}_{X,x}$, and $\mathcal{O}_X(U)$ for every subset U of X) in terms of the decomposition of f into prime factors.

Solution: Let $(f) = \prod_{i=1}^r \mathfrak{p}_i^{\nu_i}$ be the prime decomposition with distinct maximal ideals $\mathfrak{p}_i$ and all $\nu_i \geqslant 1$. Then $R/(f) \cong \bigoplus_{i=1}^r R/\mathfrak{p}_i^{\nu_i}$, and so $X := \operatorname{Spec} R/(f) \cong \coprod_{i=1}^r \operatorname{Spec} R/\mathfrak{p}_i^{\nu_i}$ by the preceding exercise. Also, each $R/\mathfrak{p}_i^{\nu_i}$ is a local ring with precisely one prime ideal $\mathfrak{p}_i/\mathfrak{p}_i^{\nu_i}$. Thus $\operatorname{Spec} R/\mathfrak{p}_i^{\nu_i}$ is a topological space with one point x_i and the stalk $R/\mathfrak{p}_i^{\nu_i}$. For any subset $U \subset X$ the sheaf property implies that $\mathcal{O}_X(U) \cong \prod_{i\colon x_i\in U} R/\mathfrak{p}_i^{\nu_i}$.

4. A scheme is *normal* if all of its local rings are integrally closed domains. Let X be an integral scheme. For each open affine subset $U = \operatorname{Spec} R$ of X, let $\tilde{R}$ be the integral closure of R in its quotient field, and let $\tilde{U} = \operatorname{Spec} \tilde{R}$. Show that one can glue the schemes $\tilde{U}$ to obtain a normal integral scheme $\tilde{X}$, called the *normalization of* X. Show also that there is a morphism $\tilde{X} \to X$ with the following universal property: for every normal integral scheme Z every morphism $f\colon Z \to X$ with $f(Z)$ dense in X factors uniquely through $\tilde{X}$.

   Solution: See Proposition 1.22 in Section 4.1 of *Algebraic Geometry and Arithmetic Curves* by Qing Liu.

5. Set $X := \operatorname{Spec} K[X,Y]$ for a field K, and let 0 denote its point corresponding to the maximal ideal $(X,Y) \subset K[X,Y]$. Show that the open subscheme $U := X \smallsetminus \{0\}$ is not affine.
   Hint: Determine $\mathcal{O}_X(U)$ using the standard open subsets D_X and D_Y.

   Solution (sketch): The subscheme U is the union of the two standard open subsets $D_X \cong \operatorname{Spec} K[X^{\pm 1}, Y]$ and $D_Y \cong \operatorname{Spec} K[X, Y^{\pm 1}]$, whose intersection is the standard open subset $D_{XY} \cong \operatorname{Spec} K[X^{\pm 1}, Y^{\pm 1}]$. By the sheaf property we deduce that $\mathcal{O}_X(U)$ is the inverse limit of the diagram

$$K[X^{\pm 1}, Y] \to K[X^{\pm 1}, Y^{\pm 1}] \leftarrow K[X, Y^{\pm 1}].$$

   As the maps are injective, the inverse limit is simply the intersection. Using the common $\mathbb{Z}^2$-grading on these rings, we quickly calculate that $\mathcal{O}_X(U) \cong K[X,Y]$. More precisely, this shows that the given restriction map induces an isomorphism $K[X,Y] = \mathcal{O}_X(X) \xrightarrow{\sim} \mathcal{O}_X(U)$. If U were affine, the equivalence of categories between affine schemes and (the opposite category of the category of) rings would imply that the inclusion morphism $U \hookrightarrow X$ is an isomorphism, which it isn't.

6. Let X be an irreducible scheme, and let $\eta \in X$ be its generic point. Prove that the intersection of all non-empty open subsets of X is $\{\eta\}$.

   Solution: By the lecture every non-empty open subset of X contains η. Conversely, by the uniqueness of η, for any $x \in X \smallsetminus \{\eta\}$ we have $\overline{\{x\}} \neq X$ and hence $\eta \notin \overline{\{x\}}$. Thus $X \smallsetminus \overline{\{x\}}$ is an open neighborhood of η that does not contain x.

*7. Let Y be an irreducible scheme with generic point η and let $f\colon X \to Y$ be a morphism of schemes. Show that the map $Z \mapsto f^{-1}(\eta) \cap Z$ is a bijective map from the set of irreducible components of X meeting $f^{-1}(\eta)$ onto the set of irreducible components of $f^{-1}(\eta)$, and the generic point of Z is the generic point of $f^{-1}(\eta) \cap Z$.

   Solution: For any irreducible component $Z \subset X$ meeting $f^{-1}(\eta)$, its image $f(Z)$ is dense in Y because it contains the generic point of Y. Let z be the generic point of Z, which exists by a proposition from the lecture because Z is irreducible. Then $f(Z) = f(\overline{\{z\}}) \subset \overline{\{f(z)\}}$, thus $Y = \overline{f(Z)} \subset \overline{\{f(z)\}} \subset Y$. Together with uniqueness of the generic point of Y, this implies that $\eta = f(z)$. Further $Z \cap$

$f^{-1}(\eta)$ is the closure of z in $f^{-1}(\eta)$ and thus irreducible. On the other hand, any irreducible subset of $f^{-1}(\eta)$ containing z must be contained in $Z \cap f^{-1}(\eta)$, hence $Z \cap f^{-1}(\eta)$ is an irreducible component of $f^{-1}(\eta)$. Since all irreducible components of $f^{-1}(\eta)$ are contained in an irreducible component of X, we have that every irreducible component Z of X meeting $f^{-1}(\eta)$ admits a generic point which lies in $f^{-1}(\eta)$; this yields the desired bijection .

8. What's wrong about [Görtz-Wedhorn, Prop.3.27 (2)]? Find 3 more mistakes in this otherwise commendable book.

   Solution: One must add that X is non-empty.

*9. (Moduli space of tuples of distinct points) Fix $n \geqslant 2$. For any ring R set

$$\begin{aligned} \mathcal{M}_n(R) &:= \big\{(x_1,\ldots,x_n) \in \mathbb{A}^n(R) \bigm| \forall i<j\colon x_i - x_j \in R^\times\big\}, \\ \overline{\mathcal{M}}_n(R) &:= \mathcal{M}_n(R)/B(R), \end{aligned}$$

where $B(R)$ is the group of invertible affine linear substitutions $x \mapsto ax+b$ with $a \in R^\times$ and $b \in R$, acting by $(x_1,\ldots,x_n) \mapsto (ax_1+b,\ldots,ax_n+b)$. Show that both $\mathcal{M}_n$ and $\overline{\mathcal{M}}_n$ extend naturally to contravariant functors from the category of affine schemes to the category of sets. Show that these are representable by certain explicit affine schemes.

Solution (sketch): The functors $\mathcal{M}_n$ and $\overline{\mathcal{M}}_n$ are represented by $\operatorname{Spec} R_n$ and $\operatorname{Spec} \overline{R}_n$ for

$$\begin{aligned} R_n &:= \mathbb{Z}[X_i|_{1\leqslant i\leqslant n}, \tfrac{1}{X_i - X_j}|_{1\leqslant i<j\leqslant n}], \\ \bar{R}_n &:= \tilde{R}_n/(X_1, X_2 - 1) \end{aligned}$$

For $\mathcal{M}_n$ this is a direct calculation. For $\overline{\mathcal{M}}_n$ first show that the functor is isomorphic to the subfunctor $\mathcal{M}'_n$ of $\mathcal{M}_n$ that is defined by the additional conditions $x_1 = 0$ and $x_2 = 1$. The reason for this is that for any tuple $(x_1,\ldots,x_n) \in \mathcal{M}_n(R)$ there exist unique $a \in R^\times$ and $b \in R$ such that $(ax_1+b,\ldots,ax_n+b) = (0,1,*,\ldots,*)$. Finally, by the same calculation as for $\mathcal{M}_n$ the subfunctor $\mathcal{M}'_n$ is representable by $\operatorname{Spec} \overline{R}_n$.