D-MATH
Prof. Richard Pink

Algebraic Geometry

FS 2017

Solutions Sheet 7

Subschemes, Finiteness Conditions

Exercises 3 and 9 are taken or adapted from *Algebraic Geometry* by Hartshorne. Exercises 2, 5, 6 and 7 are from *Algebraic Geometry I* by Görtz and Wedhorn.

1. Let k be a field, set $U_i := \mathbb{A}^1_k = \operatorname{Spec} k[X_i]$ for $i = 1, 2$ and consider the open subschemes $U_{ij} := \mathbb{A}^1_k \smallsetminus \{0\} = \operatorname{Spec} k[X_i, X_i^{-1}]$ for $i \neq j$. Let X be the scheme obtained by gluing U_1 and U_2 along U_{12} and U_{21} via $\varphi\colon U_{12} \xrightarrow{\sim} U_{21}$, $X_2 \mapsto X_1$.

   (a) Show that X is not affine.

   (b) Show that X is integral and noetherian.

   Solution: (a) If X were affine, it would be isomorphic to $\operatorname{Spec} \mathcal{O}_X(X)$. We claim that this is not the case. Similarly to the example of $\mathbb{P}^1_k$ in the lecture, we compute the ring of global sections $\mathcal{O}_X(X) = \mathcal{O}_X(U_1) \cap \mathcal{O}_X(U_2)$, where the intersection is as subrings of $\mathcal{O}_X(U_{12}) = k[X_1, X_1^{-1}]$. This yields $\mathcal{O}_X(X) = k[X_1]$ and thus $\operatorname{Spec} \mathcal{O}_X(X) = \mathbb{A}^1_k$. If X were isomorphic to $\mathbb{A}^1_k$, then there would be a bijective correspondence between closed points of X and maximal ideals of $k[X_1]$, given by taking a closed point to the ideal of polynomials which vanish at that point. But the two closed points of the 'double origin' correspond to the same ideal (X_1) of $k[X_1]$.

   (b) The scheme X is noetherian because $X = U_1 \cup U_2$ is a finite affine open covering where the coordinate rings $\mathcal{O}_X(U_i) = k[X_i]$ are both noetherian. To prove integrality, note first that $X \neq \varnothing$. Let $U \subset X$ be an arbitrary non-empty open subset. Then $U \cap U_1$ is non-empty and $\mathcal{O}_X(U) \subset \mathcal{O}_X(U \cap U_1) \cong \mathcal{O}_{\mathbb{A}^1_k}(U \cap U_1)$. As the latter is an integral domain, so is the former. Varying U we conclude that X is integral.

2. Prove that every locally closed embedding $i\colon Z \to X$ is a monomorphism in the category of schemes.

   Solution: By the definition of locally closed embeddings the underlying map of sets i is injective and for any $z \in Z$ the ring homomorphism $i^\sharp_z\colon \mathcal{O}_{X,i(z)} \to \mathcal{O}_{Z,z}$ is surjective. We claim that this alone implies that i is a monomorphism. So let $f, g\colon Y \to Z$ be two morphisms of schemes such that $i \circ f = i \circ g$. Since i is injective we obtain $f = g$ for the underlying maps. Further, for any $y \in Y$ we have $f^\sharp_y \circ i^\sharp_{f(y)} = g^\sharp_y \circ i^\sharp_{f(y)}$ for the ring homomorphisms $\mathcal{O}_{X,i(f(y))} \to \mathcal{O}_{Y,y}$. Since $i^\sharp_{f(y)}$ is surjective this implies that $f^\sharp_y = g^\sharp_y$. Varying $y \in Y$ we conclude that $f^\sharp = g^\sharp$ and thus $f = g$ as morphisms of schemes.

*3. Let $f\colon Z \to X$ be a morphism of schemes. Show that there is a unique closed subscheme Y of X with the property: the morphism f factors through Y, and if Y' is any other closed subscheme of X through which f factors, then $Y \to X$ also factors through Y'. A reasonable name for this is the *scheme-theoretic closure of the image of* f. Show further that if Z is a reduced scheme, then Y is just the reduced induced structure on the closure of the image $f(Z)$.

Solution: We first prove the statements in the affine case $X = \operatorname{Spec} R$. In this case f is given by a ring homomorphism $f^\flat\colon R \to \mathcal{O}_Z(Z)$. Then $Y := \operatorname{Spec} f^\flat(R) \cong \operatorname{Spec} R/\operatorname{Ker}(f^\flat)$ can be viewed as a closed subscheme of X with the embedding $g\colon Y \hookrightarrow X$. Also $f^\flat$ factors as $R \overset{g^\flat}{\twoheadrightarrow} f^\flat(R) \hookrightarrow \mathcal{O}_Z(Z)$ and thus f factors as $Z \to Y \overset{g}{\to} X$. Consider any other closed subscheme Y' of X through which f factors as $Z \to Y' \to X$. Since Y' is a closed subscheme of an affine scheme, it is affine and in fact $Y' = \operatorname{Spec}(R/I)$ for some ideal $I \subset R$. Thus $f^\flat$ factors as $R \to R/I \to \mathcal{O}_Z(Z)$, which implies that $I \subset \ker f^\flat = \ker g^\flat$. From this we deduce that $Y \to X$ also factors through Y'. Moreover, both $Y \hookrightarrow X$ and the factorization $Z \to Y \hookrightarrow X$ are uniquely determined by this property.

For the general case, let X be an arbitrary scheme and choose an affine open covering $X = \bigcup_{i\in I} U_i$. For all $i \in I$, the construction above defines unique closed subschemes $Y_i \subset U_i$ and factorizations $f|_{f^{-1}(U_i)}\colon f^{-1}(U_i) \to Y_i \hookrightarrow U_i$. On the overlaps, for any affine $U \subset U_i \cap U_j$, the uniqueness property ensures that $Y_i \cap U = Y_U = Y_j \cap U$. Thus we may glue the closed subschemes Y_i, $i \in I$ to a closed subscheme $Y \subset X$. In the same way, we can glue the morphisms from the universal property of each Y_i to obtain the desired morphisms for Y.

Suppose Z is a reduced scheme. Being reduced can be checked on an affine open cover, so without loss of generality suppose X is affine. Then by construction $Y = \operatorname{Spec} f^\flat(R)$ with $f^\flat(R) \hookrightarrow \mathcal{O}_Z(Z)$. Since Z is reduced, we find that $f^\flat(R)$ has no nilpotents either, so Y is reduced. The underlying map f of topological spaces factors through Y, so we have $f(Z) \subset Y$ and since Y is closed, it follows that the closure $\overline{f(Z)}$ is also contained in Y. For the reverse inclusion, suppose $Y \smallsetminus \overline{f(Z)}$ is a nonempty subset of Y. Then it contains D_s for some $s \in f^\flat(R)$, where s vanishes on all of $f(Z)$. Since $f^\flat(R) \to \mathcal{O}_Z(Z)$ is given by pulling back via $Z \to Y$, this implies that $s = 0$ in $\mathcal{O}_Z(Z)$. But then $s = 0$ in $f^\flat(R)$ as a subring of $\mathcal{O}_Z(Z)$, contradicting the assumption that D_s is a nonempty open subset of Y. Hence $\overline{f(Z)} = Y$ as desired. From the universal property of Y shown above we deduce that Y is the reduced subscheme structure on $\overline{f(Z)}$.

**4. Write out the proof of [Görtz-Wedhorn, Theorem 3.42] in all details.

5. Let X be a locally noetherian scheme. Prove that the set of irreducible components of X is locally finite, i.e. that every point of X has an open neighborhood which meets only finitely many irreducible components of X.

   Solution: By definition the irreducible components of a topological space are the maximal irreducible subspaces for the inclusion relation. By assumption any point $x \in X$ possesses an affine open neighborhood $U \subset X$ such that $\mathcal{O}_X(U)$ is noetherian. Then we already know that U has only finitely many irreducible components. It thus suffices to show that for any irreducible component Z of X the intersection $Z \cap U$ is either empty or an irreducbile component of U. So assume that $Z \cap U \neq \varnothing$. Then the same argument as in the lecture shows that $Z = \overline{Z \cap U}$ and that $Z \cap U$ is irreducible. So $Z \cap U$ is contained in some irreducible component V of U. Again by an argument in the lecture the closure $\overline{V}$ in X is irreducible, and since it contains $Z = \overline{Z \cap U}$ which is itself an irreducible component of X, we must have $Z = \overline{V}$ and hence $Z \cap U = V$, as desired.

6. Let X be a noetherian scheme. Consider the sheaf of ideals $\mathcal{N}_X$ associated to $U \mapsto \operatorname{rad}(\mathcal{O}_X(U))$, the nilradical of X. Show that $\mathcal{N}_X$ is nilpotent, i.e., there exists an integer $k \geqslant 1$ such that $\mathcal{N}_X(U)^k = 0$ for every open subset $U \subset X$.

   Solution (sketch): Let $U \subset X$ be an open subset. Since X is noetherian, there exists a nonempty open affine subset $V \subset U$ such that $\mathcal{O}_X(V)$ is noetherian and thus $\mathcal{N}_X(V)$ nilpotent by Commutative Algebra, say $\mathcal{N}_X(V)^k = 0$. Moreover, we have an injective ring homomorphism $\operatorname{res}^U_V \colon \mathcal{O}_X(U) \to \mathcal{O}_X(V)$, which maps $\mathcal{N}_X(U)^\ell$ into $\mathcal{N}_X(V)^\ell$ for all $\ell \geqslant 1$. By injectivity, we conclude that $\mathcal{N}_X(U)^k = 0$, and so we can reduce to the case where X is affine. Let $X = \bigcup_{i=1}^n U_i$ be an affine open covering with $\mathcal{O}_X(U_i)$ noetherian and $\mathcal{N}_X(U_i)^{k_i} = 0$. Set $k := \max_{i=1}^n \{k_i\}$. Then $\mathcal{N}_X(U)^k = 0$ for all open $U \subset X$ as desired.

*7. Let X be a scheme.

   (a) If X is affine, show that X^{red} is affine.

   (b) Assume that X is noetherian. If X^{red} is affine, show that X is affine.
   Hint. Use that $\mathcal{N}_X$ is nilpotent and reduce to the case $\mathcal{N}_X^2 = 0$. Then show that the canonical morphism $X \to \operatorname{Spec}\Gamma(X, \mathcal{O}_X)$ is an isomorphism.

8. Let X be a scheme over a field k. Show that

   (a) if X is locally of finite type over k, then every open covering possesses a refinement to an affine open covering $X = \bigcup_{i \in I}$ such that each $\mathcal{O}_X(U_i)$ is a finitely generated k-algebra.

   (b) X is of finite type over k if and only if X is locally of finite type over k and quasicompact.

   Solution (sketch): See proof of the analogous statement for (locally) noetherian schemes from the lecture.

9. If X is a scheme of finite type over a field, show that the set of closed points of X is dense in X. Give an example to show that this is not true for arbitrary schemes.

   Solution (sketch): We reduce first to the case where X is an affine scheme. Let $U \subset X$ be open and let $X = \bigcup_{i \in I} U_i$ with $U_i = \operatorname{Spec} R_i$ be an affine open covering of X. We claim that for any point $x \in U$, if x is closed in U, then x is closed in U_i for all U_i containing x. Indeed, pick an affine open $V = \operatorname{Spec} S$ inside $U_i \cap U$ containing x so the inclusion $U_i \cap U \hookrightarrow U_i$ gives a ring homomorphism $R_i \to S$. Both R_i and S are finitely generated k-algebras, so maximal ideals contract to maximal ideals. In particular, since x is closed in $U_i \cap U$, its image in U_i is closed. Since x is closed in each U_i and the U_i cover X, we conclude that x is closed in X. Thus, for the assertion in the exercise, it suffices to prove that every nonempty basic open subset of an affine $\operatorname{Spec} R$ contains a maximal ideal of R. If $f \in R$ is contained in every maximal ideal then f is nilpotent because R is a finitely generated k-algebra and thus the Jacobson radical is equal to the nilradical; hence D_f is empty.

   A counterexample for arbitrary schemes is the spectrum of a discrete valuation ring.