Solutions Sheet 9

Fiber Products, Schemes over Fields

Exercise 2 is from *Algebraic Geometry I* by Görtz and Wedhorn.

1. Let K be a field. Let $f \in K[X,Y,Z]$ be a nonzero homogeneous polynomial of degree $d > 0$ and $C := V(f) \subset \mathbb{P}^2_K$ the associated closed subscheme. For any nonzero $\ell \in K[X,Y,Z]$ homogeneous of degree 1 consider the associated line $L := V(\ell) \subset \mathbb{P}^2_K$. Show that if $L \not\subseteq C$, the scheme-theoretic intersection $L \cap C$ is finite of degree d over K, i.e., isomorphic to $\operatorname{Spec} R$ for some K-algebra R of dimension d over K.

   Solution: By Problem 7 of Exercise Sheet 8, the scheme-theoretic intersection is the closed subscheme associated to the graded ideal (f,ℓ) and hence isomorphic to $\operatorname{Proj} K[X,Y,Z]/(f,\ell)$. After a linear change of coordinates we may without loss of generality assume that $\ell = Z$, so that $K[X,Y,Z]/(f,\ell) \cong K[X,Y]/(g)$ with $g \in K[X,Y]$ homogeneous of degree d. Here the assumption $L \not\subseteq C$ implies that $g \neq 0$.

   If K is infinite, there are infinitely many linear polynomials in $K[X,Y]$, so after another linear change of coordinates we may assume that $X \nmid g$. Then $L \cap C$ is contained in the basic open subset $D_X \cong \operatorname{Spec} K[\frac{Y}{X}]$ of $\operatorname{Proj} K[X,Y]$ and hence isomorphic to $\operatorname{Spec} K[\frac{Y}{X}]/(\frac{g(X,Y)}{X^d}) \cong \operatorname{Spec} K[y]/(g(1,y))$. Since g is homogeneous of degree d, the assumption $X \nmid g$ implies that $\dim_K K[y]/(g(1,y)) = \deg g(1,y) = d$, as desired.

   If K is finite, reduce to the previous case by base change from $\operatorname{Spec} K$ to $\operatorname{Spec} \bar{K}$.

2. Let k be a field. Describe the fibers in all points of the following morphisms $\operatorname{Spec} B \to \operatorname{Spec} A$ corresponding in each case to the canonical homomorphism $A \to B$. Which fibers are irreducible or reduced?

   (a) $\operatorname{Spec} k[T,U]/(TU-1) \to \operatorname{Spec} k[T]$.

   (b) $\operatorname{Spec} k[T,U]/(T^2-U^2) \to \operatorname{Spec} k[T]$.

   (c) $\operatorname{Spec} k[T,U]/(T^2+U^2) \to \operatorname{Spec} k[T]$.

   (d) $\operatorname{Spec} k[T,U]/(TU) \to \operatorname{Spec} k[T]$.

   (e) $\operatorname{Spec} k[T,U,V,W]/((U+T)W, (U+T)(U^3+U^2+UV^2-V^2)) \to \operatorname{Spec} k[T]$.

   (f) $\operatorname{Spec} \mathbb{Z}[T] \to \operatorname{Spec} \mathbb{Z}$.

   (g) $\operatorname{Spec} \mathbb{Z}[T]/(T^2+1) \to \operatorname{Spec} \mathbb{Z}$.

(h) $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{Z}$.

(i) $\operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A$, where $\mathfrak{a}$ is some ideal of A.

Solution: By definition the fiber over a point $\mathfrak{p} \in \operatorname{Spec} A$ is $\operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} k(\mathfrak{p}) \cong \operatorname{Spec}(B \otimes_A k(\mathfrak{p}))$. In parts (a) to (d) we can write $B = A[U]/(f)$ for some polynomial $f \in k[T,U]$, and consequently $B \otimes_A k(\mathfrak{p}) \cong k(\mathfrak{p})[U]/(f(t,U))$, where $t \in k(\mathfrak{p})$ is the value of T at the point $\mathfrak{p}$.

(a) Here $k[T,U]/(TU-1)$ is isomorphic to the localization $k[T,T^{-1}]$ of $k[T]$, so the morphism $\operatorname{Spec} k[T,U]/(TU-1) \to \operatorname{Spec} k[T]$ is the embedding of the standard open subset obtained by removing the closed subset $V(T)$. Thus the fiber over a point $\mathfrak{p}$ is $\varnothing$ for $\mathfrak{p} = (T)$ and $\operatorname{Spec} k(\mathfrak{p})$ otherwise. In either case it is reduced, but only in the second case it is irreducible.

(b) If $\operatorname{char} k \neq 2$ and $\mathfrak{p} \neq (T)$, the two values $\pm t \in k(\mathfrak{p})$ are distinct, and by the Chinese Remainder Theorem we have $k(\mathfrak{p})[U]/(U^2 - t^2) \cong k(\mathfrak{p})[U]/(U-t) \times k(\mathfrak{p})[U]/(U+t) \cong k(\mathfrak{p})^2$, whose spectrum is reduced, but not irreducible (the underlying space is a disjoint union of two points). If $\operatorname{char} k = 2$ or $\mathfrak{p} = (T)$, we have $k(\mathfrak{p})[U]/(U^2 - t^2) \cong k(\mathfrak{p})[U]/(U-t)^2 \cong k(\mathfrak{p})[V]/(V)^2$, whose spectrum is irreducible, but not reduced, being a point with multiplicity 2.

(c) If $\operatorname{char} k = 2$ or $\mathfrak{p} = (T)$, a calculation as in (b) shows that $k(\mathfrak{p})[U]/(U^2+t^2) \cong k(\mathfrak{p})[U]/(U+t)^2 \cong k(\mathfrak{p})[V]/(V)^2$, whose spectrum is irreducible, but not reduced. Otherwise, if $k(\mathfrak{p})$ contains a solution i of the equation $X^2+1=0$, a calculation as in (b) shows that $k(\mathfrak{p})[U]/(U^2+t^2) \cong k(\mathfrak{p})[U]/(U-it) \times k(\mathfrak{p})[U]/(U+it) \cong k(\mathfrak{p})^2$, whose spectrum is reduced, but not irreducible. In the remaining case the equation $X^2+1=0$ has no solution in $k(\mathfrak{p})$; hence the polynomial U^2+t^2 is irreducible in $k(\mathfrak{p})[U]$ and $k(\mathfrak{p})[U]/(U^2+t^2)$ is a field; so the fiber is both irreducible and reduced.

(d) For $\mathfrak{p} \neq (T)$ we have $t \in k(\mathfrak{p})^\times$ and hence $k(\mathfrak{p})[U]/(tU) \cong k(\mathfrak{p})$; while for $\mathfrak{p} = (T)$ we have $t = 0$ and hence $k(\mathfrak{p})[U]/(tU) = k(\mathfrak{p})[U]$. Correspondingly the fiber is $\mathbb{A}^0_{k(\mathfrak{p})}$, respectively $\mathbb{A}^1_{k(\mathfrak{p})}$. In both cases the ring is an integral domain; so the fiber is both irreducible and reduced.

(e) Set $f := U^3+U^2+UV^2-V^2$; then $B \otimes_A k(\mathfrak{p}) \cong k(\mathfrak{p})[U,V,W]/(U+t)\cdot(W,f)$. Its spectrum is a closed subscheme of $\mathbb{A}^3_{k(\mathfrak{p})}$. For the underlying set (though not necessarily for the subscheme) we have $V((U+t)\cdot(W,f)) = V(U+t) \cup V((W,f))$. Here $V(U+t)$ is a plane parallel to the (V,W)-coordinate plane, and $V((W,f))$ is the curve defined by f within the (U,V)-coordinate plane. Neither of these is contained in the other; hence the fiber is reducible.

If $\operatorname{char} k = 2$, then $f = (U+1)(U+V)^2$ and the ring $k(\mathfrak{p})[U,V,W]/(U+t)\cdot(W,f)$ contains nilpotents. Thus, in this case, the fibers are all non-reduced.

If $\operatorname{char} k \neq 2$, then $f = (U+1)U^2 + (U-1)V^2$ is primitive as a polynomial in V over $k[U]$ because $(U+1)U^2$ and $U-1$ are coprime, and irreducible over $k(U)$

because $-(U+1)U^2/(U-1)$ has no square root in $k(U)$. Thus f is irreducible in $k[U,V] \cong k[U,V,W]/(W)$; hence (W,f) is a prime ideal. In particular it is its own radical. For any value of t, check that the product $(U+t)(W,f)$ is equal to the intersection $(U+t) \cap (W,f)$. It follows that the ideal $(U+t)(W,f)$ is radical: $\mathrm{Rad}((U+t)(W,f)) = \mathrm{Rad}(U+t) \cap \mathrm{Rad}(W,f) = (U+t) \cap (W,f) = (U+t)(W,f)$, and thus the ring $k(\mathfrak{p})[U,V,W]/(U+t)\cdot(W,f)$ contains no nilpotents and all fibers are reduced.

(f) Here $\operatorname{Spec} \mathbb{Z}[T] = \mathbb{A}^1_{\mathbb{Z}}$; hence for any $\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}$ the fiber is $\operatorname{Spec} k(\mathfrak{p})[T] = \mathbb{A}^1_{k(\mathfrak{p})}$, which is both irreducible and reduced.

(g) By the same arguments as in (c) the ring $B \otimes_A k(\mathfrak{p}) \cong k(\mathfrak{p})[T]/(T^2+1)$ is

$$\begin{cases} \cong k(\mathfrak{p})[U]/(U^2) & \text{if } T^2+1=0 \text{ has a double solution in } k(\mathfrak{p}), \\ \cong k(\mathfrak{p})^2 & \text{if } T^2+1=0 \text{ has two distinct solutions in } k(\mathfrak{p}), \\ \text{a field} & \text{if } T^2+1=0 \text{ has no solution in } k(\mathfrak{p}). \end{cases}$$

Here the first case occurs if $\mathfrak{p} = (2)$, the second if $\mathfrak{p} = (p)$ for a prime $p \equiv 1\ (4)$, and the third for all other prime ideals (because $\mathbb{F}_p^\times$ has order $p-1$ and $\sqrt{-1} \notin \mathbb{Q}$). Accordingly, the fiber is irreducible but not reduced, respectively reduced but not irreducible, respectively irreducible and reduced.

(h) The fiber over the generic point (0) has the coordinate ring $\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{C}$ and is therefore irreducible and reduced. For any prime p we have $\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} = 0$; hence the fiber is empty and thus reduced, but not irreducible.

(i) For any $\mathfrak{p} \in \operatorname{Spec} A$, we have $k(\mathfrak{p}) = S^{-1}(A/\mathfrak{p})$ with $S := (A/\mathfrak{p}) \setminus \{0\}$, and hence

$$A/\mathfrak{a} \otimes_A k(\mathfrak{p}) \;\cong\; S^{-1}(A/\mathfrak{a} \otimes_A A/\mathfrak{p}) \;\cong\; S^{-1}A/(\mathfrak{a}+\mathfrak{p}).$$

In the case $\mathfrak{a} \subset \mathfrak{p}$ this is simply $k(\mathfrak{p})$ again. Otherwise the image of S in $A/(\mathfrak{a}+\mathfrak{p})$ contains 0; hence the ring is zero.

3. Let k be a field. Consider $X := \mathbb{A}^2_k = \operatorname{Spec} k[X,Y]$ and $\mathbb{P}^1_k = \operatorname{Proj} k[T,U]$. Let $\tilde{X}$ denote the closed subscheme of $X \times_k \mathbb{P}^1_k$ defined by the equation $XU = TY$. Determine the irreducible components and the fibers of $\tilde{X}$ and $\tilde{X} \times_X \tilde{X}$ over X.

   Solution (sketch): Let $O \in X$ denote the origin and set $U := X \setminus \{O\}$. Then the morphism $\pi \colon \tilde{X} \to X$ induces an isomorphism $\pi^{-1}(U) \to U$. Thus for any point $x \in U$, the fiber over x is $\operatorname{Spec} k(x)$. The fiber over O is a copy of the projective line $\mathbb{P}^1_k$. Also $\tilde{X}$ is irreducible. For a detailed explanation as well as a proof of the last statement, see for example Hartshorne, p. 28.

   Since fiber product commutes with taking fibers, any fiber of $\tilde{X} \times_X \tilde{X} \to X$ is the product of two copies of the fiber of $\tilde{X} \to X$, which is therefore again $\operatorname{Spec} k(x)$, respectively $\mathbb{P}^1_k \times \mathbb{P}^1_k$. Also $\tilde{X} \times_X \tilde{X}$ is the union of the image of the diagonal embedding of $\tilde{X}$ and the fiber $\mathbb{P}^1_k \times \mathbb{P}^1_k$ over O. Both of these are irreducible of dimension 2, and none of them is contained in the other; hence these are the irreducible components of $\tilde{X} \times_X \tilde{X}$.