Introduction to Interpolation Theory

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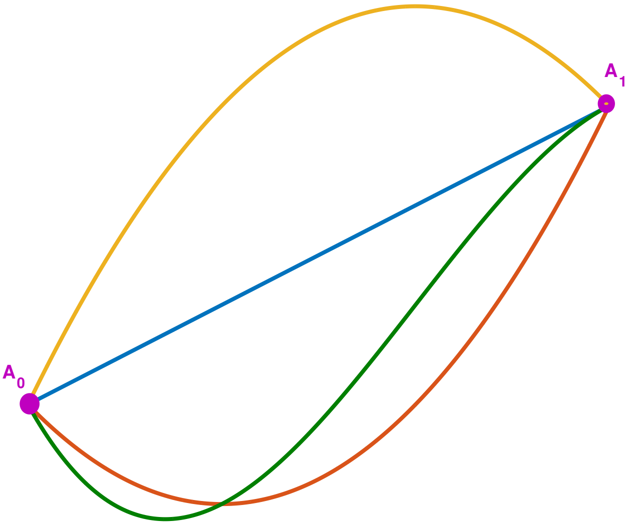
April,6 2017

Reminder 1: Interpolation between two data points



Reminder 2: Marcinkiewicz interpolation theorem

Theorem (Marcinkiewicz interpolation theorem)

Assume that T is a linear operator with the following properties $(p_0 \neq p_1)$

i) $T : L^{p_0} \to L^{q_0,\infty} \quad \text{with} \quad \text{norm} \quad M_0$

ii) $T : L^{p_1} \to L^{q_1,\infty} \quad \text{with} \quad \text{norm} \quad M_1$

put

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

and assume that $p \leq q$.

Then

$$T : L^p \to L^q$$

with norm

$$M \leq C_\theta M_0^{1-\theta} M_1^{\theta}.$$

Remarks

- important point: "extension of the linear operator T"
  $\rightarrow$ interpolation property (property of the operator)

- extension to quasilinear operators
  $\rightarrow$ application to the maximal function

- generalization to $T : L^{p_i}(U, d\mu) \rightarrow L^{q_i}(V, d\nu)$

Reminder 3: Interpolation inequality

Assume that $f \in L^p \cap L^q$ with $1 \leq p \leq q \leq \infty$.
Then $f \in L^r$ for all r such that $p \leq r \leq q$ with norm

$$||f||_r \leq ||f||_p^{1-\theta} ||f||_q^{\theta}.$$

where

$$\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$$

$\rightarrow$ interpolation inequality (property of the function)

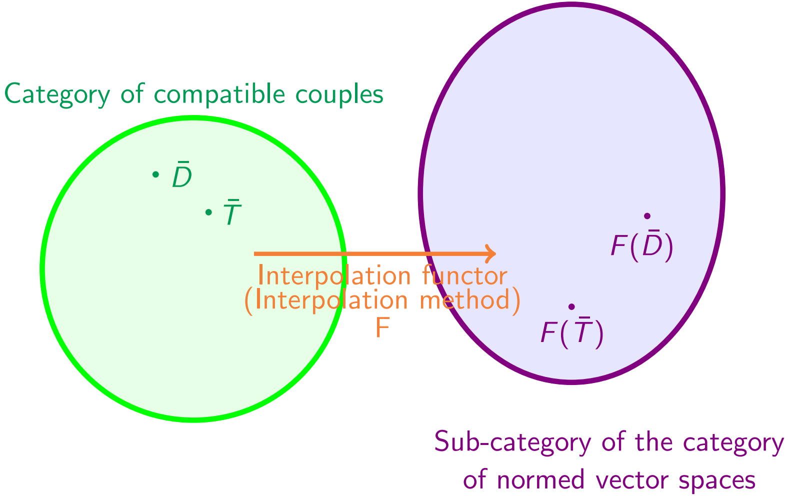
Interpolation theory in a nutshell

Let A_i, $i = 0, 1$, be subspaces of a Hausdorff topological vector space $\mathcal{A}$, and let B_i, $i = 0, 1$, be subspaces of a Hausdorff topological vector space $\mathcal{B}$.
Assume in addition that T is a bounded linear operator from A_i to B_i.
Interpolation is then the search for pairs of spaces $(A, B) \in \mathcal{A} \times \mathcal{B}$ such that

$$T : A \to B$$

is still a bounded linear operator.

Abstract context



Some references

- J. Bergh, J. Löfström; Interpolation space - an introduction; Springer
- A. Lunardi; Interoilation theory; Edizioni Della Normale (Pisa)
- H. Triebel; Interpolation theory, function spaces, differential operators; North-Holland
- L. Tartar; An introduction to Sobolev spaces and interpolation spaces; Springer