Interpolation Theory

1) Introduction → see slides!

Recall: •) A Hausdorff topological vector space is a topological vector space such that any two distinct points can be separated.

•) A category consists of objects A, B, C and morphisms $T, R, S \dots$ between objects such that

if $T: A \to B$ and $S: B \to C$ then $\exists$ morphism ST the product ~~[illegible]~~ such that $ST: A \to C$ with the property $T(SR) = (TS)R$.

and $\forall A$ $\exists$ morphism $I = I_A$ such that $\forall T: A \to A$:
$TI = IT = A$

•) A subcategory S of a category $\mathcal{C}$ consists of a subcollection of the objects of $\mathcal{C}$, denotes $ob(S)$ and a " of " morphisms of $\mathcal{C}$, " $hom(S)$ such that

- $\forall$ X in $ob(S)$: $I_A \in hom(S)$
- $\forall$ $T: X \to Y \in hom(S)$: $X, Y \in ob(S)$
- $\forall$ T, S in $hom(S)$: $TS \in hom(S)$

•) A functor is a mapping from category $\mathcal{A}$ to category $\mathcal{B}$ which assigns to each object in $\mathcal{A}$ an object in $\mathcal{B}$
" " " morphism " " " morphism "

such that $F(I_A) = I_{F(A)}$

$$F(TS) = F(T)F(S)$$

1) Introduction $\Rightarrow$ slides

2) Basic notions

Def 1. Let A_0 and A_1 be two topological vector spaces.
Then A_0 and A_1 are called compatible if there exists a Hausdorff topological vector space $\mathcal{V}$ such that A_0 and A_0 are subspaces of $\mathcal{V}$. In this case we write $\bar{A} = (A_0, A_1)$

$\mathcal{V}$ is just needed to be able to construct $A_0 + A_1 = \Sigma(\bar{A})$.

Rmk. 2 In the above context, we can form

i) the sum $\Sigma(\bar{A}) := A_0 + A_1 = \{ a = a_0 + a_1 \; ; \; a_0 \in A_0, \; a_1 \in A_1 \}$

ii) the intersection $\Delta(\bar{A}) := A_0 \cap A_1 = \{ a \mid a \in A_0 \cap A_1 \}$
$= \{ a \mid a \in A_0 \wedge a \in A_1 \}$

and in addition we can endow $\Sigma(\bar{A})$ and $\Delta(\bar{A})$ with the following norms:

i) $\|a\|_{\Sigma(\bar{A})} := \inf\limits_{a = a_0 + a_1} \left(\|a_0\|_{A_0} + \|a_1\|_{A_1} \right)$

ii) $\|a\|_{\Delta(\bar{A})} := \max \{ \|a\|_{A_0}, \|a\|_{A_1} \}$

Lemma 3 Let A_0 and A_1 be compatible normed vector spaces.
Then
i) $A_0 \cap A_1 = \Delta(\bar{A})$ with the above norm is a normed vector space

ii) $A_0 + A_1 = \Sigma(\bar{A})$ with the above norm is a normed vector space

iii) If A_0 and A_1 are both complete, then also $\Delta(\bar{A})$ and $\Sigma(\bar{A})$ are complete.

$\rightarrow$ Proof: Exercise / Bergh-Löfström p. 24

Rmk. / Def. 4 In what follows we will work with ~~[illegible]~~ the following important categories (loosely speaking think about "sets")

most often $\mathcal{C} = \mathcal{N}$ or $\mathcal{C} = \mathcal{B}$

α) $\mathcal{N}$ the category of all normed vector space and $\mathcal{C}$ any sub-category of $\mathcal{N}$, e.g. $\mathcal{B}$ the category of all Banach spaces.

β) The category $\mathcal{C}_1$ of all compatible couples,

→ objects: $\bar{A} = (A_0, A_1)$ such that A_0 and A_1 are compatible and such that $A_0, A_1, \Sigma(\bar{A})$ and $\Delta(\bar{A})$ are objects in $\mathcal{C}$

→ morphisms: $T : \bar{A} \longrightarrow \bar{B}$ are the bounded linear maps (operators)

$$T : \Sigma(\bar{A}) \longrightarrow \Sigma(\bar{B})$$

such that $T\big|_{A_i} : A_i \longrightarrow B_i \quad i = 0, 1$

are morphisms in $\mathcal{C}$, i.e. linear bounded maps.

Def. 5 Let $\bar{A} = (A_0, A_1)$ be a couple in $\mathcal{C}_1$, i.e. a compatible couple. Then A space in $\mathcal{C}$ is called an intermediate space between A_0 and A_1 (or: with respect to $\bar{A}$) if

$$\Delta(\bar{A}) \subset A \subset \Sigma(\bar{A}) \text{ with continuous inclusions.}$$

Def. 6 Let $\bar{A}$ and $\bar{B}$ be two couples in $\mathcal{C}_1$.
Then two spaces A and B in $\mathcal{C}$ are called interpolation spaces with respect to $\bar{A}$ and $\bar{B}$ if

i) A is an intermediate space w.r. to $\bar{A}$
ii) B " " " " " " " $\bar{B}$
iii) $T : \bar{A} \to \bar{B}$ bounded linear $\Rightarrow$ $T : A \to B$ bounded linear
i.e.
$T : A_i \to B_i$, $i = 0,1$ bounded linear $\Rightarrow$ $T : A \to B$ bounded lin.

Def. 7 An interpolation functor (an interpolation method) is a functor $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}$ such that
if $\bar{A}, \bar{B}$ are couples in $\mathcal{C}_1$
then $\mathcal{F}(\bar{A})$ and $\mathcal{F}(\bar{B})$ are interpolation spaces with respect to $\bar{A}$ and $\bar{B}$.
Moreover, we require $\mathcal{F}(T) = T$ for all $T : \bar{A} \to \bar{B}$

This is exactely the required interpolation property!
As we will see, an ~~the~~ interpolation property implies a corresponding interpolation inequality!
→ look at the operator norms

3) Example of an interpolation method: The K-method (real interpolation)

In the following, assume that X and Y are Banach spaces. (*)

Def. 8 Let (X,Y) be a compatible couple, i.e. $(X,Y)\in \mathcal{C}_1$.
Then, $\forall x\in X+Y$ and $t>0$, set

$$K(t,x,X,Y) = K(t,x) \text{ if there is no confusion}$$
$$= \inf_{\substack{x=a+b\\ a\in X\\ b\in Y}} \left(\|a\|_X + t\|b\|_Y\right)$$

Rmk. 9 i) Note that $K(1,x) = \|x\|_{X+Y}$

ii) Note that $\forall t>0$ fixed $t\|b\|_Y$ is an equivalent norm to $\|b\|_Y$

iii) Observe that $\forall t>0$ $K(t,\cdot)$ is a norm on $X+Y$ which is equivalent to the norm of $X+Y$.

Def. 10 Let $0<\theta<1$, $1\le p\le\infty$ and set

$$(X,Y)_{\theta,p} = \left\{ x\in X+Y : t\mapsto t^{-\theta}K(t,x) \in L^q\left(\mathbb{R}^+, \frac{dt}{t}\right)\right\} \quad (*)$$

with norm $\underbrace{\|x\|_{(X,Y)_{\theta,p}}}_{=\|x\|_{\theta,p} \text{ if there is no confusion}} = \|t^{-\theta}K(t,x)\|_{L^q(\mathbb{R}^+,\frac{dt}{t})}$

I.e. $(X,Y)_{\theta,p} = \mathcal{F}((X,Y))$ where the functor $\mathcal{F}$ is given by (*).

As we will see, this functor is an interpolation method according to our definition!

(*) It is possible to work in the larger context of $\mathcal{N}$, the category of normed vector spaces ($\to$ cf. Bergh – Löfström)

Rmk. 11 i) Check that in fact $\|\cdot\|_{\theta,p}$ is (in fact) a norm.

ii) If $X = Y$, $X + Y = X$ and $K(t,x) \le \min\{1,t\}\|x\|_X$.

From that, we deduce that

$$(X,X)_{\theta,p} = X \quad , 0<\theta<1 \, , \, 1 \le p \le \infty$$

with equivalence of norms.

iii) Note that $K(t,x;X,Y) = t\,K(t^{-1},x;Y,X)$ $\forall t > 0$.

And by $\tau = t^{-1}$ we get

$$(X,Y)_{\theta,p} = (Y,X)_{1-\theta,\, p} \qquad 0<\theta<1, \; 1 \le p \le \infty$$

So, the order is relevant!

Rmk. iv) Assume that $Y \subset X$, (X,Y) a compatible couple.

Then, it holds $\forall x \in X$

$$K(t,x) = \inf_{\substack{x=a+b \\ a \in X,\, b \in Y}} \left(\|a\|_X + t\|b\|_Y \right) \leq \|x\|_X \qquad \text{take } a=x,\ b=0$$

$$\Rightarrow \quad \forall a>0 : \quad \left(\int_a^\infty t^{-\theta p} K(t,x)^p \frac{dt}{t} \right)^{1/p}$$

$$\leq \|x\|_X \left(\int_a^\infty t^{-\theta p} \frac{dt}{t} \right)^{1/p}$$

$$= \|x\|_X \left(t^{-\theta p} \frac{(-1)}{\theta p} \Big|_a^\infty \right)$$

$$= \|x\| \frac{1}{\theta p} a^{-\theta p}$$

Thus, only the behaviour near the origin is crucial.

Actually one could replace the real half line in the definition of $(X,Y)_{\theta,p}$ by any interval $(0,a)$ with equivalent norms.

Prop. 12 For $0<\theta<1$ and $(1\le p\le\infty)$ $1\le p_1\le p_2\le\infty$ we have

$$X\cap Y\subset (X,Y)_{\theta,p_1}\subset (X,Y)_{\theta,p_2}\subset (X,Y)_{\theta,\infty}\subset X+Y$$

This shows that $(X,Y)_{\theta,p}$ are intermediate spaces with respect to (X,Y).

Proof: <u>Step 1 : $(X,Y)_{\theta,\infty}\subset X+Y$</u>

Observe that by definition, $\|x\|_{X+Y}=K(1,x)$.
Therefore we have

$$\|x\|_{X+Y}=K(1,x)\le\|x\|_{\theta,\infty}\qquad\forall x\in(X,Y)_{\theta,\infty}$$

This shows that $(X,Y)_{\theta,\infty}$ is continuously embedded in $X+Y$.

<u>Step 2 : $(X,Y)_{\theta,p}\subset(X,Y)_{\theta,\infty}$</u>

For each $x\in(X,Y)_{\theta,p}$ and $t>0$ observe that $K(\cdot,x)$ is increasing.
Thus, we get

$$t^{-\theta}K(t,x)=(\theta p)^{1/p}\Big(\underbrace{\int_t^\infty s^{-\theta p-1}\,ds}_{=-\frac{1}{\theta p}s^{-\theta p}\Big|_{s=t}^{\infty}=\frac{1}{\theta p}t^{-\theta p}}\Big)^{1/p}K(t,x)$$

$$\le(\theta p)^{1/p}\Big(\int_t^\infty s^{-\theta p-1}K(s,x)^p\,ds\Big)^{1/p}\qquad t>0$$

$$=(\theta p)^{1/p}\Big(\int_t^\infty s^{-\theta p}K(s,x)^p\frac{ds}{s}\Big)^{1/p}\le(\theta p)^{1/p}\Big(\int_0^\infty s^{-\theta p}K(s,x)^p\frac{ds}{s}\Big)^{1/p}$$

$$=(\theta p)^{1/p}\|x\|_{\theta,p}$$

From this, we decude that $x \in (X,Y)_{\theta,\infty}$ with $\|x\|_{\theta,\infty} \le C\,\|x\|_{\theta,p}$

Remark: a refinement of this argument shows that actually

$$\|x\|_{\theta,\infty} \le \left(\min\{\theta,1-\theta\}\cdot p\right)^{1/p} \|x\|_{\theta,p} \qquad \boxed{(*)}$$

Namely: Observe that $K(t,x) \le \frac{t}{s} K(s,x)$ $\quad x \in X+Y,\ 0<s<t$

Therefore: $$t^{1-\theta} K(t,x) = [(1-\theta)p]^{1/p} \Big(\underbrace{\int_0^t s^{(1-\theta)p-1}\, ds}_{= \frac{1}{(1-\theta)p} s^{1-\theta}\Big|_0^t = \frac{1}{(1-\theta)p} t^{1-\theta}}\Big)^{1/p} K(t,x)$$

$$\le [(1-\theta)p]^{1/p} \left(\int_0^t s^{-\theta p-1}\, t^{p}\, K(s,x)^p\right)^{1/p}$$

$$\Rightarrow t^{-\theta} K(t,x) \le [(1-\theta)p]^{1/p} \left(\int_0^t s^{-\theta p-1} K(s,x)^p\, ds\right)^{1/p}$$

$$\le [(1-\theta)p]^{1/p} \underbrace{\left(\int_0^\infty s^{-\theta p} K(s,x)^p \frac{ds}{s}\right)^{1/p}}_{= \|x\|_{\theta,p}} \quad \Rightarrow \quad \boxed{(*)}$$

Step 3 : $(X,Y)_{\theta,p_1} \subset (X,Y)_{\theta,p_2}$ for $p_1 < p_2$

Let $x \in (X,Y)_{\theta,p_1}$. Then we have

$$\|x\|_{\theta,p_2} = \left(\int_0^\infty t^{-\theta p_2} K(t,x)^{p_2} \frac{dt}{t}\right)^{1/p_2}$$

$$= \left(\int_0^\infty t^{-\theta p_1} K(t,x)^{p_1} \left(t^{-\theta} K(t,x)\right)^{(p_2-p_1)} \frac{dt}{t}\right)^{1/p_2}$$

$$\leq \left(\int_0^\infty t^{-\theta p_1} K(t,x)^{p_1} \frac{dt}{t}\right)^{1/p_2} \cdot \left(\sup_{t>0} t^{-\theta} K(t,x)\right)^{(p_2-p_1)/p_2}$$

$$= \left(\|x\|_{\theta,p_1}\right)^{p_1/p_2} \left(\|x\|_{\theta,\infty}\right)^{1-p_1/p_2}$$

Using finally the result from step 2 we get

$$\|x\|_{\theta,p_2} \leq C \|x\|_{\theta,p_1}^{p_1/p_2} \cdot \|x\|_{\theta,p_1}^{1-p_1/p_2} = C \|x\|_{\theta,p_1}$$

since $\|x\|_{\theta,\infty} \leq C \|x\|_{\theta,p_1} \Rightarrow \|x\|_{\theta,\infty}^{1-p_1/p_2} \leq C \|x\|_{\theta,p_1}^{1-p_1/p_2}$

Step 4 : $X \cap Y \subset (X,Y)_{\theta,p}$ $\quad 0<\theta<1, \quad 1 \leq p \leq \infty$

Observe that $K(t,x) \leq \min\{1,t\} \|x\|_{X \cap Y}$ for $x \in X \cap Y$.

From that, we immediately deduce :

$$\|x\|_{\theta,p} = \left(\int_0^\infty t^{-\theta p} K(t,x)^p \frac{dt}{t}\right)^{1/p}$$

$$\leq \left(\int_0^\infty t^{-\theta p} \min\{1,t\}^p \, t^{-1} \|x\|_{X\cap Y}^p \, dt\right)^{1/p}$$

$$\leq \|x\|_{X\cap Y} \left(\int_0^\infty t^{-\theta p - 1} (\min\{1,t\})^p \, dt\right)^{1/p} \leq \dots \leq C \|x\|_{X \cap Y}$$

split the integral $\int_0^1 + \int_1^\infty$ and use elementary calculus

Prop. 13 For all $\theta \in (0,1)$, $p \in [1, \infty]$
$(X,Y)_{\theta,p}$ is a Banach space.

(Recall that we assumed that X and Y are Banach spaces.)

Proof: Let $\{x_n\}$ be a Cauchy sequence in $(X,Y)_{\theta,p}$.

Due to the continuous embedding $(X,Y)_{\theta,p} \subset X+Y$ (see Prop. 12)
$\{x_n\}$ is also a Cauchy sequence in $X+Y$.
Sine we assumed X and Y to be complete,
$\{x_n\}$ converges to some $x \in X+Y$ (cf. also Lemma 3)

Next, we estimate $\|x_n - x\|_{\theta,p}$:
Fix $\varepsilon > 0$ such that $\|x_n - x_m\|_{\theta,p} \leq \varepsilon \quad \forall\, n,m \geq n_\varepsilon$.
Moreover, sine $K(t,y)$ is a norm on $X+Y$ we have

(*) $$K(t, x_n - x) \leq K(t, x_n - x_m) + K(t, x_m - x) \quad \forall\, n,m,\ t>0$$

From this last inequality we deduce

(#) $$t^{-\theta} K(t, x_n - x) \leq t^{-\theta} K(t, x_n - x_m) + t^{-\theta} \max\{t,1\} \|x_m - x\|_{X+Y}$$

multiply (*) by $t^{-\theta}$ and observe that

$$\begin{aligned}
K(t, x_m - x) &= \inf_{\substack{x_m - x = a+b \\ a\in X,\, b\in Y}} \left(\|a\|_X + t\|b\|_Y\right) \\
&\leq \inf_{\substack{x_m - x = a+b \\ a\in X,\, b\in Y}} \left(\max\{1,t\}\|a\|_X + \max\{1,t\}\|b\|_Y\right) \\
&= \max\{1,t\} \inf_{\substack{x_m - x = a+b \\ a\in X,\, b\in Y}} \left(\|a\|_X + \|b\|_Y\right) \\
&= \max\{1,t\} \|x_m - x\|_{X+Y}
\end{aligned}$$

In what follows we will distinct two cases: $p = \infty$ and $p < \infty$:

Case 1: $p = \infty$

For $t > 0$ and $n, m \geq n_\varepsilon$ we have

$$t^{-\theta} K(t, x_n - x) \leq \varepsilon + t^{-\theta} \max\{t, 1\} \| x_m - x \|_{X+Y}$$

since $\| x_n - x_m \|_{\theta,\infty} = \sup_t t^{-\theta} K(t, x_n - x_m, X, Y)$)

$$\Rightarrow \quad t^{-\theta} K(t, x_n - x_m) \leq \| x_n - x_m \|_{\theta,\infty} \leq \varepsilon \quad \text{by assumption}$$

Letting $m \to \infty$ we deduce that

$$t^{-\theta} K(t, x_n - x) \leq \varepsilon \quad \forall t > 0$$

and thus $x_n \to x$ in $(X, Y)_{\theta,\infty}$ with $x \in (X, Y)_{\theta,\infty}$.

This shows that $(X, Y)_{\theta,\infty}$ is complete.

Case 2: $p < \infty$

We start from the following observation:

$$\| x_n - x \|_{\theta,\infty} = \left(\int_0^\infty t^{-\theta p} K(t, x_n - x)^p \frac{dt}{t} \right)^{1/p}$$

$$= \lim_{\delta \to 0} \left(\int_\delta^{1/\delta} t^{-\theta p} K(t, x_n - x)^p \frac{dt}{t} \right)^{1/p}$$

Moreover, from $(*)$ we get for $n, m \geq n_\varepsilon$

$$\left(\int_\delta^{1/\delta} t^{-\theta p} K(t, x_n - x)^p \frac{dt}{t} \right)^{1/p} \leq \left(\int_\delta^{1/\delta} t^{-\theta p} K(t, x_n - x_m)^p \frac{dt}{t} \right)^{1/p}$$

$$+ \| x_m - x \|_{X+Y} \left(\int_\delta^{1/\delta} t^{-\theta p} (\max\{1, t\})^p \frac{dt}{t} \right)^{1/p}$$

(apply to both sides of (#): $\left(\int_\delta^{1/\delta} (\cdots)^p \frac{1}{t} \, dt \right)^{1/p}$)

in particular: rhs $\left(\int_\delta^{1/\delta} \| x_m - x \|_{X+Y}^p \, t^{-\theta p} (\max\{1, t\})^p \frac{1}{t} \, dt \right)^{1/p}$

$$\leq \|x_n - x_m\|_{\theta,p} + \|x_m - x\|_{X+Y} \left(\int_\delta^{1/\delta} t^{-\theta p} \max\{1,t\} \frac{dt}{t} \right)^{1/p}$$

by assumption

$$\leq \varepsilon + \|x_m - x\|_{X+Y} \underbrace{C(\delta, p, \theta)}_{\text{may become big}}$$

since $\left(\int_\delta^{1/\delta} t^{-\theta p} (\max\{1,t\})^p \frac{dt}{t} \right)^{1/p} = C(p, \delta, \theta)$

e.g. $p=2$ $\quad \int_\delta^{1/\delta} t^{-\theta \cdot 2} \max\{1,t\}^2 \, t^{-1} \, dt$

$$= \int_\delta^1 t^{-2\theta-1} \, dt + \int_1^{1/\delta} t^{-2\theta+2-1} \, dt$$

$$= \frac{1}{-2\theta} \, t^{-2\theta} \Big|_\delta^1 + t^{-2\theta+p} \frac{1}{2-2\theta} \Big|_1^{1/\delta}$$

$$= \frac{-1}{2\theta} + \frac{1}{2\theta} \delta^{-2\theta} + \delta^{-p+2\theta} \frac{1}{2-2\theta} - \frac{1}{2-2\theta}$$

Finally let $m \to \infty$ and then $\delta \to 0$.

↳ i.e. such that $\|x_m - x\|_{X+Y}$ forces $\|x_m - x\|_{X+Y} \, C(\delta, p, \theta)$ to tend to zero

Thus at the end we get that

$$x_n \to x \text{ in } (X,Y)_{\theta,p} \text{ with } x \in (X,Y)_{\theta,p}$$

Theorem 14 Let (X_0, X_1) and (Y_0, Y_1) be two compatible couples.
If $T \in L(X_0, Y_0) \cap L(X_1, Y_1)$ then
$T \in L((X_0, X_1)_{\theta,p}, (Y_0, Y_1)_{\theta,p}) \quad \forall\, 0 < \theta < 1,\ 1 \le p \le \infty$.

Moreover;

$$\|T\|_{L((X_0,X_1)_{\theta,p},\,(Y_0,Y_1)_{\theta,p})}$$
$$\le \left(\|T\|_{L(X_0,Y_0)}\right)^{1-\theta} \left(\|T\|_{L(X_1,Y_0)}\right)^{\theta}$$

This is exactly the interpolation property we are looking for! Together with Prop. 12 this shows tha K is really an interpolation functor! (take into accout that K does not "affect" morphisms)

Proof: Step 1: Assume that $\|T\|_{L(X_0,Y_0)} \neq 0$

Let $x \in (X_0, X_1)_{\theta,p}$ and let $x = a + b$, $a \in X_0$, $b \in X_1$ be a representation of x.

Then, for arbitrary $t > 0$ we have

$$\|Ta\|_{Y_0} + t\|Tb\|_{Y_1} \le \|T\|_{L(X_0,Y_0)} \left(\|a\|_{X_0} + t \frac{\|T\|_{L(X_1,Y_1)}}{\|T\|_{L(X_0,Y_0)}} \|b\|_{X_1} \right)$$

since $\|Ta\|_{Y_0} \le \|T\|_{L(X_0,Y_0)} \|a\|_{X_0}$

and

$$\|Tb\|_{Y_1} \le \|T\|_{L(X_1,Y_1)} \|b\|_{X_1} = \|T\|_{L(X_0,Y_0)} \frac{\|T\|_{L(X_1,Y_1)}}{\|T\|_{L(X_0,Y_0)}} \|b\|_{X_1}$$

fine since $\|T\|_{L(X_0,Y_0)} \neq 0$ by assumption

From this last inequality we deduce that

$$K(t,Tx,Y_0,Y_1) \le \|T\|_{L(X_0,Y_0)} \, K\left(t \frac{\|T\|_{L(X_1,Y_1)}}{\|T\|_{L(X_0,Y_0)}}, x, X_0, X_1\right). \quad (*)$$

Next, we set $s = t\,\frac{\|T\|_{L(X_1,Y_1)}}{\|T\|_{L(X_0,Y_0)}}$ and calculate

$$\|Tx\|_{(Y_0,Y_1)_{\theta,p}} = \left(\int_0^\infty t^{-\theta p} K(t,Tx,Y_0,Y_1)^p \frac{dt}{t}\right)^{1/p}$$

$$\le \|T\|_{L(X_0,Y_0)} \left(\int_0^\infty t^{-\theta p} K\left(t\frac{\|T\|_{L(X_1,Y_1)}}{\|T\|_{L(X_0,Y_0)}}, x, X_0, X_1\right)^p \frac{dt}{t}\right)^{1/p}$$

due to $(*)$

$$= \|T\|_{L(X_0,Y_0)} \left(\int_0^\infty \frac{\|T\|^{-\theta p}_{L(X_0,Y_0)}}{\|T\|^{-\theta p}_{L(X_1,Y_1)}} s^{-\theta p} K(s,x,X_0,X_1)^p \frac{\|T\|_{L(X_1,Y_1)}}{\|T\|_{L(X_0,Y_0)}} \frac{1}{s} \frac{\|T\|_{L(X_0,Y_0)}}{\|T\|_{L(X_1,Y_1)}} ds\right)^{1/p}$$

by the change of variables $s = t\,\frac{\|T\|_{L(X_1,Y_1)}}{\|T\|_{L(X_0,Y_0)}}$

$$= \|T\|_{L(X_0,Y_0)} \frac{\|T\|^{\theta}_{L(X_1,Y_1)}}{\|T\|^{\theta}_{L(X_0,Y_0)}} \underbrace{\|x\|_{\theta,p}}_{=\|x\|_{(X_0,X_1),\theta,p}}$$

$$= \|T\|^{1-\theta}_{L(X_0,Y_0)} \cdot \|T\|^{\theta}_{L(X_1,Y_1)} \|x\|_{(X_0,X_1)_{\theta,p}}$$

From this, we immediately deduce that

$$\|T\|_{L((X_0,X_1)_{\theta,p},\,(Y_0,Y_1)_{\theta,p})} \le \|T\|^{1-\theta}_{L(X_0,Y_0)} \cdot \|T\|^{\theta}_{L(X_1,Y_1)}$$

Step 2: $\|T\|_{L(X_0,Y_0)} = 0$

Replace $\|T\|_{L(X_0,Y_0)}$ everywhere by ε, perform the same observations as in step 1 and let finally tend ε to zero in order to conclude.

Corollary 15 Let (X,Y) be a compatible couple.
For $0<\theta<1$ and $1\leq p\leq\infty$ there exists a constant $C=C(\theta,p)$ such that

$$\|y\|_{(X,Y)_{\theta,p}} \leq C(\theta,p)\,\|y\|_X^{1-\theta}\,\|y\|_Y^{\theta} \quad \forall\, y\in X\cap Y.$$

This ist exactly the interpolation inequality we are looking for!

Proof: Set $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$ in dependence of whether X and Y are real or complex Banach spaces.

For $y\in X\cap Y$ we define the following linear operator

$$T:\ \mathbb{K}\longrightarrow X\cap Y$$
$$\lambda\longmapsto\lambda y$$

Then clearly $\|T\|_{L(\mathbb{K},X)}=\|y\|_X$ and

$\|T\|_{L(\mathbb{K},Y)}=\|y\|_Y$ and

$\|T\|_{L(\mathbb{K}=(\mathbb{K},\mathbb{K})_{\theta,p},\,(X,Y)_{\theta,p})}=\|y\|_{(X,Y)_{\theta,p}}$

The assertion then immediately follows from Thm. 14.

4) Examples

Ex 16 For $0<\theta<1$, [redacted] we have

$$(C(\mathbb{R}^n), C^1(\mathbb{R}^n))_{\theta,\infty} = C^\theta(\mathbb{R}^n).$$

with equivalence of norms.

Proof: „$\subseteq$"

Let $f \in (C(\mathbb{R}^n), C^1(\mathbb{R}^n))_{\theta,\infty}$ and pick $x \neq y \in \mathbb{R}^n$.

Furthermore let $f = a+b$ be a decomposition of f with $a \in C(\mathbb{R}^n)$ and $b \in C^1(\mathbb{R}^n)$.

First of all, observe that $\|f\|_\infty \leq \|a\|_\infty + \|b\|_\infty$.
This immediately implies

$$\|f\|_\infty \leq \underbrace{K(1, f, C(\mathbb{R}^n), C^1(\mathbb{R}^n))}_{\substack{= \inf_{f=a+b} (\|a\|_\infty + \|b\|_{C^1}) \\ = \inf_{f=a+b} (\|a\|_\infty + \|b\|_\infty + \|D_1 f\|_\infty)}} \leq \underset{\sup t^{-\theta} K(t,f)}{\|f\|_{\theta,\infty}} \qquad (*)$$

Moreover, we can estimate

$$|f(x)-f(y)| \leq |a(x)-a(y)| + |b(x)-b(y)|$$
$$\leq 2\|a\|_\infty + \|b\|_{C^1}\, |x-y| \leq 2\|a\|_\infty + 2\|b\|_{C^1}\, |x-y|$$

which implies that

$$|f(x)-f(y)| \leq 2K(|x-y|, f) \leq 2|x-y|^\theta \|f\|_{\theta,\infty}$$

$$\|f\|_{\theta,\infty} = \sup t^{-\theta} K(t,f)$$

$$\Rightarrow K(|x-y|, f) \leq |x-y|^\theta \|f\|_{\theta,\infty}$$

This implies that $\frac{|f(x)-f(y)|}{|x-y|^\theta} \leq 2\|f\|_{\theta,\infty}$
which finally - together with $(*)$ implies that $f \in C^\theta(\mathbb{R}^n)$.

"2"

$\rightsquigarrow$ We have to obtain a suitable decomposition
$f = a + b$

So, let $f \in C^\theta$.

Let in addition φ be a smooth, compactly supported function (assume $\operatorname{supp} \varphi \subset B_1(0)$) such that $\int_{\mathbb{R}^n} \varphi(x)dx = 1$.

Then, for $t > 0$ set for $x \in \mathbb{R}^n$

$$b_t(x) = \frac{1}{t^n} \int_{\mathbb{R}^n} f(y)\, \varphi\left(\frac{x-y}{t}\right) dy$$

and

$$a_t(x) = f(x) - b_t(x).$$

Obviously, in this setting we have $f = a_t + b_t$!

Next, we look at $a_t(x)$:

We have that $a_t(x) = \frac{1}{t^n} \int_{\mathbb{R}^n} \big(f(x) - f(x-y)\big)\, \varphi\left(\frac{y}{t}\right) dy$

use $\int_{\mathbb{R}^n} \varphi(y)dy = 1$, the def. of a_t and b_t

and a change of variables in the def. of b_t

from that we can estimate

$$\|a_t\|_\infty \leq \underbrace{[f]_{C^\theta}}_{\sup \frac{|f(x)-f(y)|}{|x-y|^\theta}} \frac{1}{t^n} \int_{\mathbb{R}^n} |y|^\theta \varphi\left(\frac{y}{t}\right) dy = t^\theta [f]_{C^\theta} \int_{\mathbb{R}^n} |w|^\theta \varphi(w)\, dw$$

$\frac{y}{t} = w$
$dy = t^n dw$

Then, we look at $b_t(x)$:

Obviously, we have $\|b_t\|_\infty \leq \|f\|_\infty$ follows immediately from the def. of b_t and the fact that $\int_{\mathbb{R}^n} \varphi(z)\,dz = 1$

Moreover, $D_i b_t(x) = \frac{1}{t^{n+1}} \int_{\mathbb{R}^n} f(y) D_i \varphi\left(\frac{x-y}{t}\right) dy$,

from which we deduce

$$D_i b_t(x) = \frac{1}{t^{n+1}} \int_{\mathbb{R}^n} \left(f(x-y) - f(x)\right) D_i \varphi\left(\frac{y}{t}\right) dy$$

because $\int_{\mathbb{R}^n} D_i \varphi\left(\frac{x-y}{t}\right) dy = 0$ since φ is compactly supported

Finally we find $\|D_i b_t\|_\infty \leq t^{\theta-1} [f]_{C^\theta} \int_{\mathbb{R}^n} |w|^\theta D_i \varphi(w)\,dw$ (as before)

which implies

$$t^{-\theta} K(t,f) \leq t^{-\theta}\left(\|a_t\|_\infty + t\|b_t\|_{C^1}\right) \leq$$

$$\leq t^{-\theta}\left(t^\theta C[f]_{C^\theta} + t \cdot t^{\theta-1} C[f]_{C^\theta}\right) + C\|f\|_\infty$$

$$= C\left([f]_{C^\theta} + \|f\|_\infty\right) \leq C\|f\|_{C^\theta}$$

Details !

which concludes the proof.

$$t^{-\theta} K(t,f) = t^{-\theta} \inf_{\substack{f=a+b \\ a\in C^0 \\ b\in C^1}} \left(\|a\|_{C^0} + t\|b\|_{C^1} \right) \le t^{-\theta} \left(\|a_t\|_\infty + t\|b\|_\infty + t\|D_i b\|_\infty \right)$$

$$\begin{aligned}
\|f\|_{\theta,\infty} &= \sup_t t^{-\theta} K(t,f) \\
&= \|a_t + b_t\|_{\theta,\infty} \le \|a_t\|_{\theta,\infty} + \|b_t\|_{\theta,\infty} \\
&\qquad = \sup_t t^{-\theta} K(t,a_t) + \sup_t t^{-\theta} K(t,b_t) \\
&\qquad = \sup_t t^{-\theta} \inf_{\substack{a_t = x+y \\ x\in C \\ y\in C^1}} \left(\|x\|_C + t\|y\|_{C^1} \right) \\
&\qquad\quad + \sup_t t^{-\theta} \inf_{\substack{b_t = r+s \\ r\in C \\ s\in C^1}} \left(\|r\|_C + t\|s\|_{C^1} \right) \\
&\le \sup_t t^{-\theta} \|a_t\|_\infty + \sup_t t^{-\theta} \inf_{b_t = r+s} \left(\|r\|_C + t\|s\|_{C^1} \right)
\end{aligned}$$

by setting $x = a_t$, $y = 0$

$$\le \sup_t t^{-\theta} t^{\theta} [f]_\theta + \sup_t t^{-\theta} \min\{1,t\} \|b_t\|_{C\cap C^1}$$ (see above p. 5)

$$\begin{aligned}
&\le [f]_\theta + \sup_t t^{-\theta} \min\{1,t\} \left(\|b_t\|_\infty + \|D_i b_t\|_\infty \right) \\
&\le [f]_\theta + \sup_t t^{-\theta} \min\{1,t\} \|b_t\|_\infty + \sup_t t^{-\theta} t^{\theta-1} C [f]_\theta \min\{1,t\} \\
&\le [f]_\theta + C\|f\|_\infty + \sup_{t\le 1} t^{-1} \min\{1,t\} C [f]_\theta + \sup_{t\ge 1} t^{-\theta} t^{\theta-1} C [f]_\theta \max\{1,t\} \\
&\le C\|f\|_\infty + C[f]_\theta \le C \|f\|_{C^\theta}
\end{aligned}$$

Ex 17 For $0<\theta<1$ and $1 \le p < \infty$ we have

$$\left(L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n)\right)_{\theta,p} = W^{\theta,p}(\mathbb{R}^n)$$

where $W^{\theta,p}$ is the Slobodeckij space of all $f \in L^p$ such that

$$[f]_{W^{\theta,p}} = \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\theta p}} \, dx dy \right)^{1/p} < \infty$$

Rmk. 18 $W^{\theta,p} = B^{\theta}_{p,p} = F^{\theta}_{p,p}$

Sobolev (Morrey): $f \in W^{1,p}(\mathbb{R}^n) \underset{p>n}{\Longrightarrow} |f(x)-f(y)| \le C|x-y|^{\alpha} \|\nabla f\|_p$

and $C^{\alpha} = B^{\alpha}_{\infty,\infty}$, $\alpha = 1 - \frac{n}{p}$

recall also: $B^{\theta}_{p,p} \subset C$ if $\theta > \frac{n}{p}$

and

$B^{\theta}_{p,p} \subset B^{\theta-\varepsilon}_{p,\infty} \subset B^{\theta-\varepsilon-\frac{n}{p}}_{\infty,\infty}$

$\theta - \varepsilon - \frac{n}{p} = s - \frac{n}{\infty}$

Here, also cases $\boxed{p \neq n}$ are covered!

Proof: „$\subseteq$"

Let $f \in (L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n))$, in particular assume that we have a representation $f = a + b$, with $a \in L^p$, $b \in W^{1,p}$.

Then we can estimate

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x+h) - f(x)|^p}{|h|^{n+p\theta}} \, dx\, dh$$

$$\leq C(p) \int_{\mathbb{R}^n \times \mathbb{R}^n} \left(\frac{|a(x+h) - a(x)|^p}{|h|^{n+p\theta}} + \frac{|b(x+h) - b(x)|^p}{|h|^{n+p\theta}} \right) dx\, dh$$

$$\leq C(p) \int_{\mathbb{R}^n} \frac{\|a\|_p^p}{|h|^{n+p\theta}} \, dh + C(p) \int_{\mathbb{R}^n} \frac{|h|^p \, \|Db\|_p^p}{|h|^{n+p\theta}} \, dh$$

since for all fixed $h \neq 0$ we have

$$\left(\int_{\mathbb{R}^n} \frac{|b(x+h) - b(x)|^p}{|h|^p} \, dx \right)^{1/p} \leq \|Db\|_{L^p}$$

$$\left(\leq C(p) \left[\|a\|_p^p \int_{\mathbb{R}^n} |h|^{-n-\theta p} + \|b\|_{W^{1,p}} \int_{\mathbb{R}^n} |h|^{p-n-\theta p} \, dh \right] \right)$$

$$= C(p) \int_{\mathbb{R}^n} |h|^{-n-\theta p} \big(\underbrace{\|a\|_p + |h| \, \|b\|_{W^{1,p}}} \big)^p \, dh$$

$$= C(p) \int_{\mathbb{R}^n} |h|^{-n-\theta p} \, K(|h|, f)^p \, dh$$

$$= C(p) \int_0^\infty \frac{K(r,f)^p}{r^{n+\theta p}} \cdot r^{n-1} \int_{S^{n-1}} d\sigma \, dr = C(p,n) \int_0^\infty r^{-\theta p} K(r,f)^p \frac{dr}{r}$$

$$= C(p,n) \, \|f\|_{\theta,p} \qquad (1)$$

Moreover, since $X+Y=X=L^p(\mathbb{R}^n)$ in our present case and since $(L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n))_{\theta,p} \subset L^p(\mathbb{R}^n)$ (see Prop. 12) we have

$$\|f\|_p \leq C\|f\|_{\theta,p} \qquad 2)$$

So, finally (1) and (2) together imply that $(L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n)) \subset W^{\theta,p}(\mathbb{R}^n)$ as claimed.

„2"

Again, we have to construct a suitable decomposition $f=a+b$.
In fact, as in Example 16 we set

$$b_t(x) = \frac{1}{t^n}\int_{\mathbb{R}^n} f(y)\,\varphi\left(\frac{x-y}{t}\right)dy \quad , \ \varphi: \text{compactly supported, smooth, pos.}$$

$$\int_{\mathbb{R}^n}\varphi(z)\,dz = 1 \ , \ \operatorname{supp}\varphi \subset B_1(0)$$

and

$$a_t(x) = f(x) - b_t(x)$$

And as in the previous example, we estimate a_t and b_t separately in the suitable norms

Estimate of a_t

$$\|a_t\|_p^p = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |f(y)-f(x)|\,\frac{1}{t^n}\varphi\left(\frac{x-y}{t}\right)dy\right)^p dx$$

(same reason as above)

$$\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(y)-f(x)|^p\,\frac{1}{t^n}\varphi\left(\frac{x-y}{t}\right)dy\,dx$$

by Jensen's inequality for the probability measure $t^{-n}\varphi\left(\frac{x-y}{t}\right)dy$.

Using this, we get

$$\int_0^\infty t^{-\theta p}\,\|a_t\|_p^p\,\frac{dt}{t} \leq \int_0^\infty\int_{\mathbb{R}^n\times\mathbb{R}^n} t^{-\theta p}\,|f(y)-f(x)|^p\,\frac{1}{t^n}\varphi\left(\frac{x-y}{t}\right)dy\,dx\,dt$$

%.

$$= \int_{\mathbb{R}^n\times\mathbb{R}^n} |f(y)-f(x)|^p \int_0^\infty t^{-\theta p} \frac{1}{t^n} \varphi\left(\frac{x-y}{t}\right) \frac{dt}{t} \, dx\,dy$$

$$\leq \int_{\mathbb{R}^n\times\mathbb{R}^n} |f(y)-f(x)|^p \int_{|x-y|}^\infty t^{-\theta p} \frac{1}{t^n} \varphi\left(\frac{x-y}{t}\right) \frac{dt}{t} \, dx\,dy$$

due to the fact that φ is compactly supported in the unit ball

$$\leq \frac{\|\varphi\|_\infty}{\theta p+n} \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(y)-f(x)|^p}{|y-x|^{\theta p+n}} \, dx\,dy$$

$$\int_{|x-y|}^\infty t^{-\theta p-n-1} \, dt = t^{-\theta p-n} \frac{(-1)}{\theta p+n} \Big|_{|x-y|}^\infty = \frac{1}{\theta p+n} |x-y|^{-\theta p-n}$$

$$= C(p,n) \, [f]^p_{W^{\theta,p}} \qquad (1)$$

Estimate of b_t

i) $\|b_t\|_p^p \leq \int_{\mathbb{R}^n} \frac{1}{t^n} \int_{\mathbb{R}^n} |f(y)|^p \varphi\left(\frac{x-y}{t}\right) dy\,dx$ (by Jensen's inequality)

$$\leq \|f\|_p^p \underbrace{\|\varphi\|_1}_{=1} \implies \|b_t\|_p \leq \|f\|_p$$

ii) $D_i b_t(x) = \frac{1}{t^{n+1}} \int_{\mathbb{R}^n} (f(x-y)-f(x)) \, D_i\varphi\left(\frac{x-y}{t}\right) dy$ (as before)

From this, we infer

$$\int_0^\infty t^{(1-\theta)p} \|D_i b_t\|_p^p \frac{dt}{t} \leq \int_0^\infty t^{(1-\theta)p} \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{1}{t^{(n+1)p}} |f(x-y)-f(x)|^p \left|D_i\varphi\left(\frac{x-y}{t}\right)\right|^p dy\,dx \frac{dt}{t}$$

$$\leq \int_{|x-y|}^\infty t^{-\theta p-np-1} \int_{\mathbb{R}^n\times\mathbb{R}^n} |f(x-y)-f(x)|^p \left|D_i\varphi\left(\frac{x-y}{t}\right)\right|^p dy\,dx\,dt$$

$$\le C \int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x-y)-f(x)|^p \, \|D_i \varphi\|_\infty^p \, \frac{1}{|x-y|^{\theta p + n}} \, dx\,dy$$

$$= \|D_i\varphi\|_\infty^p \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x-y)-f(x)|^p}{|x-y|^{n+\theta p}} \, dx\,dy = \|D_i \varphi\|_\infty^p \, [f]^p_{W^{\theta,p}} = C\,[f]^p_{W^{\theta,p}} \qquad (2)$$

Conclusion

$$\|f\|_{\theta,p} = \left(\int_0^\infty t^{-\theta p} K(t,f)^p \frac{dt}{t} \right)^{1/p} = \left(\int_0^\infty t^{-\theta p} \Big[\inf_{\substack{f=a+b \\ a \in L^p \\ b \in W^{1,p}}} \left(\|a\|_p + t \|b\|_{W^{1,p}} \right) \Big]^p \frac{dt}{t} \right)^{1/p}$$

$$\le \left(\int_0^\infty t^{-\theta p} \left(K(t, a_t, L^p, W^{1,p}) \right)^p \frac{dt}{t} \right)^{1/p} + \left(\int_0^\infty t^{-\theta p} K(t, b_t, L^p, W^{1,p})^p \frac{dt}{t} \right)^{1/p}$$

$$\le \left(\int_0^\infty \left(t^{-\theta p} \|a_t\|_p^p \frac{dt}{t} \right)^{1/p} + \int_0^\infty t^{-\theta p} \left(\min\{1,t\} \|b_t\|_{L^p \cap W^{1,p}} \right)^p \frac{dt}{t} \right)^{1/p}$$

$$\le \underset{\text{by } (1)}{C\,[f]_{W^{\theta,p}}} + \int_0^\infty t^{-\theta p} (\min\{1,t\})^p \left(\|b_t\|_p^p + \|D_i b_t\|_p^p \right)^{p/p} \frac{dt}{t} \Bigg)^{1/p}$$

$$= C[f]_{W^{\theta,p}} + \left(\int_0^1 t^{-\theta p} t^p t^{-1} \|b_t\|_p^p \, dt + \int_1^\infty t^{-\theta p} \|b_t\|_p^p t^{-1} \, dt \right.$$

$$\left. + \int_0^\infty t^{(1-\theta)p} \|D_i b_t\|_p^p \frac{dt}{t} \right)^{1/p}$$

$$\le C[f]_{W^{\theta,p}} + \left(\|b_t\|_p^p C(p,\theta) + \|b_t\|_p^p C(p,\theta) + C[f]^p_{W^{\theta,p}} \right)^{1/p} \quad \text{by } (2)$$

$$\le C \|f\|_{W^{\theta,p}}$$ This completes the proof.

$$K(t, a_t, L^p, W^{1,p}) = \inf_{\substack{a_t = x+y \\ x \in L^p \\ y \in W^{1,p}}} \left(\|x\|_p + \|y\|_{W^{1,p}} \cdot t \right) \le \|a_t\|_p$$

and : $K(t, b_t, L^p, W^{1,p}) \le \min\{1,t\} \overbrace{\|b_t\|_{L^p \cap W^{1,p}}}^{= \max\{\|b_t\|_p, \, \|b_t\|_{W^{1,p}}\} = \left(\|b_t\|_p^p + \|D_i b_t\|_p^p\right)^{1/p}}$: see p. 5

Ex 18 Let $0<\theta<1$ and $1\leq q\leq\infty$ and let (Ω,μ) be a σ-finite measure space.
Then, $\left(L^1(\Omega), L^\infty(\Omega)\right)_{\theta,q} = L^{\frac{1}{1-\theta},q}(\Omega)$

Rmk 19 i) Let $\mathcal{V}$ be the space of all measurabe, a.e. finite functions on Ω.
Then, $\mathcal{V}$ is obviously a linear topological Hausdorff vector space under convergence in measure on each measurabe set $E\subset\Omega$ with $\mu(E)<\infty$.
Since $L^1(\Omega)$ and $L^\infty(\Omega)$ are both continuously embedded in $\mathcal{V}$ $(L^1(\Omega), L^\infty(\Omega))$ is a compatible couple.

ii) Using further results („Reiteration Theorem"), one can actually prove: for $1\leq p_1<p_2\leq\infty$

$$\left(L^{p_1}(\Omega), L^{p_2}(\Omega)\right)_{\theta,q} = L^{p,q}(\Omega) \text{ with}$$

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$$

(→ see e.g. Adams, Fournier; Sobolev spaces)

iii) Reminder about Lorentz spaces:
Recall that for $1<p\leq\infty$, $1\leq q\leq\infty$ the norms $\|f\|_{L^{p,q}}$ and $|f|_{L^{p,q}}$ are equivalent where

$$\|f\|_{L^{p,q}} = \begin{cases} \left(\int_0^\infty \left(t^{1/p} f^{**}(t)\right)^q \frac{dt}{t}\right)^{1/q}, & 1\leq q<\infty \\ \sup_{t>0} t^{1/p} f^{**}(t), & q=\infty \end{cases}$$

./.

and $\quad |f|_{L^{p,q}} = \begin{cases} \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q} &, \ 1 \leq q < \infty \\ \sup_{t>0} t^{1/p} f^*(t) &, \ q = \infty \end{cases}$

where f^* is the non-increasing rearrangement of f,

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds$$

Recall also : $\quad \lambda_f(t) = \mu\{ x \in \Omega \ / \ |f(x)| > t \}$

Then, $\quad f^*(t) = \inf \{ s : \ \lambda_f(s) \leq t \}$

v) Note that with $L^{\frac{1}{1-\theta}, q}$ we remain in the range $1 < p < \infty$ (since $0 < \theta < 1$) where $\|f\|_{p,q}$ and $|f|_{p,q}$ are equivalent!

Proof of EX 18

Claim: For $t>0$ it holds

$$K(t,f,L^1,L^\infty) = \int_0^t f^*(s)\,ds = t f^{**}(t) \qquad (*)$$

Proof of $\quad (L^1(\Omega), L^\infty(\Omega))_{\theta,q} = L^{\frac{1}{1-\theta},q}$

admitting the claim:

"$\subseteq$" Let $f \in L^1 + L^\infty$.

Then we have for $q<\infty$

$$\|f\|^q_{L^{\frac{1}{1-\theta},q}}$$

$$= \int_0^\infty (t^{1-\theta} f^{**}(t))^q \frac{dt}{t} = \int_0^\infty t^{q-q\theta} f^{**}(t)^q \frac{dt}{t}$$

$$\underset{\text{by }(*)}{=} \int_0^\infty t^{-\theta q} K(t,f,L^1,L^\infty)^q \frac{dt}{t} = \|f\|^q_{\theta,q},$$

respectively for $q=\infty$

$$\|f\|_{L^{\frac{1}{1-\theta},\infty}} = \sup_{t>0} t^{1-\theta} f^{**}(t) \underset{\text{by }(*)}{=} \sup_{t>0} t^{-\theta} K(t,f,L^1,L^\infty)$$

„2"   For $f \in L^{\frac{1}{1-\theta},q}$ we estimate

in the case $q < \infty$

$$\int_0^\infty \left(t^{-\theta} K(t,f)\right)^q \frac{dt}{t} \underset{\text{by }(*)}{=} \int_0^\infty t^{-\theta q}\, t^q f^{**}(t)^q \frac{dt}{t}$$

$$= \int_0^\infty \left(t^{1-\theta} f^{**}(t)\right)^q \frac{dt}{t} = \|f\|_{L^{\frac{1}{1-\theta}},q}$$

and in the case $q = \infty$

$$\sup_{t>0} t^{-\theta} K(t,f,L^1,L^\infty) \underset{\text{by }(*)}{=} \sup_{t>0} t^{-\theta} t f^{**}(t) = \sup_{t>0} t^{1-\theta} f^{**}(t)$$

$$= \|f\|_{L^{\frac{1}{1-\theta}},q}$$

Note that what we have seen shows that we do not only have equivalence of the norms $\|f\|_{(L^1,L^\infty)_{\frac{1}{1-\theta},q}}$ and $\|f\|_{L^{\frac{1}{1-\theta}},q}$

but even equality!

Proof of the claim

" $\leq$ " $\leftarrow$ "

Let $f \in L^1 + L^\infty$ and let $x \in \Omega$.
Moreover, set

$$a(x) = \begin{cases} f(x) - f^*(t)\dfrac{f(x)}{|f(x)|} & , \text{ if } |f(x)| > f^*(t) \\ 0 & , \text{ otherwise} \end{cases}$$

$\rightsquigarrow$ recall the decomp. in Ex 16/17: $f = a_t + b_t$

and

$$b(x) = f(x) - a(x)$$

Estimate for a

Observe that
$|a(x)| = |f(x)| - f^*(t)$ if $|f(x)| > f^*(t)$ and
$|a(x)| = 0$ if $|f(x)| \leq f^*(t)$

$f(x) - f^*(t) = |f(x)| - f^*(t)$ if $f(x) > f^*(t)$

$$|a(x)| = \left| f(x) - f^*(t)\frac{f(x)}{|f(x)|} \right| = \left| -f(x) - f^*\frac{(-f(x))}{|f(x)|} \right| = |f(x)| - f^*(t)$$

if $f(x) < 0$ and $|f(x)| > f^*(t)$

Then we have $\|a\|_{L^1} = \int\limits_{E} (|f(x)| - f^*(t))\, \mu(dx)$

$E = \{x \in \Omega \,|\, |f(x)| > f^*(t)\}$ $\quad \underbrace{\lambda_f(f^*(t))}_{=|E|} = \lambda_{f^*(t)}(f^*(t))$

f and f^* are equi-measurable

Now, recall that due to the definition of f^* and $\lambda_f(t)$
we have that $\mu(\{x \in \Omega \,|\, |f(x)| > f^*(t)\}) \leq t$
(fundamental property of f^* (see e.g. Adams, Fournier; Sobolev sp.)
and f^* is constant on $[\mu(E), t]$ (" " " " ")

Thus, we get finally:

$$\|a\|_{L^1} = \int\limits_E (|f(x)| - f^*(t))\, \mu(dx) = \int_0^{\mu(E)} (f^*(s) - f^*(t))ds \leq \int_0^t (f^*(s) - f^*(t))ds$$

recall $\displaystyle\int\limits_{M} |u| = \int_0^{\mu(M)} u^*(s)ds$ $\qquad (*)$

$M = \{x \,|\, |u(x)| > s\}$

Estimate for b Moreover, again from the definition of b, we have

$$|b(x)| = \begin{cases} |f(x)| & , \text{ if } |f(x)| \leq f^*(t) \\ f^*(t) & , \text{ if } |f(x)| > f^*(t) \end{cases}$$

Thus, $|b(x)| \leq f^*(t) = \frac{1}{t}\int_0^t f^*(t)\,ds \quad , x \in \Omega$

Conclusion All together, we finally find

$$\begin{aligned} K(t,f,L^1,L^\infty) &= \inf_{\substack{f=u+v \\ u\in L^1 \\ v\in L^\infty}} \left(\|u\|_1 + t\|v\|_\infty\right) \\ &\leq \|a\|_1 + t\|b\|_\infty \\ &\leq \int_0^t \left(f^*(s) - f^*(t)\right)ds + t\cdot\frac{1}{t}\int_0^t f^*(t)\,ds \\ &= \int_0^t f^*(s)\,ds - t\cdot f^*(t) + t f^*(t) = \int_0^t f^*(s)\,ds \end{aligned}$$

„$\geq$" Let $f = a + b$ be any decomposition of f.

Then it holds:

$$f^*(s) \leq a^*((1-\varepsilon)s) + b^*(\varepsilon s) \qquad s \geq 0 \,,\ 0 < \varepsilon < 1$$

Thus, if $a \in L^1$ and $b \in L^\infty$, we get

$$\int_0^t f^*(s)\,ds \leq \int_0^t a^*((1-\varepsilon)s)\,ds + \int_0^t b^*(\varepsilon s)\,ds$$

$$\leq \underbrace{\frac{1}{1-\varepsilon} \int_0^\infty a^*(r)\,dr}_{(1-\varepsilon)s = r} + \underbrace{t \cdot b^*(0)}_{b^*\text{: non-increasing}}$$

$$\leq \frac{1}{1-\varepsilon} \int_\Omega |a|(x)\,\mu(dx) + t\,\|b\|_\infty$$

by (*) and the definition of $b^*(0)$

Letting $\varepsilon \to 0$ we get

$$\int_0^t f^*(s)\,ds \leq \|a\|_1 + t\,\|b\|_\infty \qquad (\#)$$

This finally leads to

$$K(t,f,L^1,L^\infty) = \inf_{\substack{f = u+v \\ u \in L^1 \\ v \in L^\infty}} \left(\|u\|_1 + t\,\|\ \|_\infty \right) \geq \int_0^t f^*(s)\,ds$$

because holds for any decomposition $f = a + b$, $a \in L^1$, $b \in L^\infty$

This concludes the proof and shows $K(t,f) = \int_0^t f^*(s)\,ds \underset{\text{by def.}}{=} t f^{**}(s)$.

Rmk. 20 i) An alternative proof of the Claim in Ex 18
can be found in Adams, Fournier, Sobolev spaces, p. 225 f.

ii) The decomposition $f = a + b$ in „$\leq$" gives rise to
a decomposition $L^1 + L^\infty$ of $f \in L^{pq}$