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Brownian Motion and Stochastic Calculus

Exercise sheet 1

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than March 3rd

Exercise 1.1 Let W be a Brownian motion on $[0,1]$ and define the *Brownian bridge* $X = (X_t)_{0\leq t\leq 1}$ by $X_t = W_t - tW_1$.

(a) Show that X is a Gaussian process and calculate its mean and covariance functions. Sketch a typical path of X.

(b) Show that X does **not** have independent increments.

Exercise 1.2 Let $(\Omega, \mathcal{F}, P)$ be a probability space and assume that $X = (X_t)_{t\geq 0}$, $Y = (Y_t)_{t\geq 0}$ are two stochastic processes on $(\Omega, \mathcal{F}, P)$. Two processes Z and Z' on $(\Omega, \mathcal{F}, P)$ are said to be *modifications* of each other if $P(Z_t = Z'_t) = 1 \; \forall \; t \geq 0$, while Z and Z' are *indistinguishable* if $P(Z_t = Z'_t \; \forall \; t \geq 0) = 1$.

(a) Assume that X and Y are both right-continuous or both left-continuous. Show that the processes are modifications of each other if and only if they are indistinguishable.

Remark: A stochastic process is said to *have the path property* $\mathcal{P}$ ($\mathcal{P}$ can be continuity, right-continuity, differentiability, ...) if the property $\mathcal{P}$ holds for P-almost every path.

(b) Give an example showing that one of the implications of part **a)** does not hold for general X, Y.

Exercise 1.3 Let $X = (X_t)_{t\geq 0}$ be a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. The aim of this exercise is to show the following chain of implications:

$$X \text{ optional} \Rightarrow X \text{ progressively measurable} \Rightarrow X \text{ product-measurable and adapted.}$$

(a) Show that every progressively measurable process is product-measurable and adapted.

(b) Assume that X is adapted and *every* path of X is right-continuous. Show that X is progressively measurable.
Remark: The same conclusion holds true if every path of X is left-continuous.
Hint: For fixed $t \geq 0$, consider an approximating sequence of processes Y^n on $\Omega \times [0,t]$ given by $Y^n_0 = X_0$ and $Y^n_u = \sum_{k=0}^{2^n-1} 1_{(tk2^{-n}, t(k+1)2^{-n}]}(u) X_{t(k+1)2^{-n}}$ for $u \in (0,t]$.

(c) Recall that the optional σ-field $\mathcal{O}$ is generated by the class $\overline{\mathcal{M}}$ of all adapted processes whose paths are all RCLL. Show that $\mathcal{O}$ is also generated by the subclass $\mathcal{M}$ of all *bounded* processes in $\overline{\mathcal{M}}$.

(d) Use the monotone class theorem to show that every optional process is progressively measurable.