Brownian Motion and Stochastic Calculus

Exercise sheet 10

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than May 12th

Exercise 10.1 Let $(B_t)_{t\geq 0}$ be a Brownian motion and let $(X_t)_{t\geq 0}$ be defined by $X_t = \int_0^t \operatorname{sign}(B_s)\,dB_s$, where $\operatorname{sign}(x) = 1$ for $x \geq 0$ and $\operatorname{sign}(x) = -1$ for $x < 0$.

(a) Show that $(X_t)_{t\geq 0}$ is a Brownian motion and that $E[X_t\,B_s] = 0$ for all $s, t \geq 0$ (which means that X and B are uncorrelated).

(b) Show that $E[X_t B_t^2] = 2^{\frac{5}{2}}\, t^{\frac{3}{2}}\, \frac{1}{3\sqrt{\pi}}$ and conclude that $(X_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are not independent (despite being uncorrelated and Gaussian processes).

Exercise 10.2 The goal of this exercise is to prove that *"Brownian motion does not hit points whenever $d \geq 2$"*. Let $d \geq 2$, $\Omega = C([0,\infty); \mathbb{R}^d)$ and $Y = (Y_t)_{t\geq 0}$ denote the canonical process. For each $x \in \mathbb{R}^d$, let $\mathbb{P}_x$ be the unique probability measure on $(\Omega, \mathcal{Y}^0_\infty)$ under which Y is a (d-dimensional) Brownian motion started at x.

(a) Let $0 \neq x \in \mathbb{R}^d$ and $a > 0$ such that $0 < a < |x|$ and consider the stopping time

$$\tau_{a,b} := \inf\left\{t \geq 0 \,\middle|\, |Y_t| \leq a \text{ or } |Y_t| \geq b\right\}.$$

For $d \geq 3$, show that $(X_t)_{t\geq 0}$ defined by $X_t := \left|Y_{\tau_{a,b}\wedge t}\right|^{2-d}$ is a bounded martingale under $\mathbb{P}_x$. Additionally, when $d = 2$ show that $X_t = \ln(|Y_{\tau_{a,b}\wedge t})$ is also a bounded martingale.

(b) Show that

$$\text{for any } 0 \neq x \in \mathbb{R}^d, \text{ we have } \mathbb{P}_x\left[Y_t \neq 0 \text{ for all } t \geq 0\right] = 1.$$

Exercise 10.3 Let B be a Brownian motion in $\mathbb{R}^3$, $0 \neq x \in \mathbb{R}^3$ and define the process $M = (M_t)_{t\geq 0}$ by

$$M_t = \frac{1}{|x + B_t|}.$$

This is well defined since one can show that $P[B_t = -x \text{ for some } t \geq 0] = 0$.

(a) Show that M is a continuous local martingale.
Hint: Use Itô's formula.

Moreover, show that M is bounded in L^2, i.e., $\sup_{t\geq 0} E[|M_t|^2] < \infty$.
Hint: For any $t \geq 0$, show that

$$E\left[|M_t|^2 1_{\{|M_t| \geq \frac{2}{|x|}\}}\right] = (2\pi t)^{-\frac{3}{2}} \int\limits_{|y|\leq \frac{|x|}{2}} \frac{1}{|y|^2} \exp\left(-\frac{|y-x|^2}{2t}\right) dy$$

and estimate the right-hand side from above using the reverse triangle inequality.

(b) Show that M is a *strict local martingale*, i.e., M is not a martingale.
Hint: Show that $E[M_t] \to 0$ as $t \to \infty$. To this end, similarly to part a), compute $E[M_t]$ and use the reverse triangle inequality as a first estimate. Then compute the resulting integral using spherical coordinates.
Remark: This is the standard example of a local martingale which is not a (true) martingale. It also shows that even good integrability properties like boundedness in L^2 are not enough to guarantee the martingale property.