Brownian Motion and Stochastic Calculus

Exercise sheet 10

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than May 12th

Exercise 10.1 Let $(B_t)_{t\geq 0}$ be a Brownian motion and let $(X_t)_{t\geq 0}$ be defined by $X_t = \int_0^t \operatorname{sign}(B_s)\, dB_s$, where $\operatorname{sign}(x) = 1$ for $x \geq 0$ and $\operatorname{sign}(x) = -1$ for $x < 0$.

(a) Show that $(X_t)_{t\geq 0}$ is a Brownian motion and that $E[X_t\, B_s] = 0$ for all $s, t \geq 0$ (which means that X and B are uncorrelated).

(b) Show that $E[X_t B_t^2] = 2^{\frac{5}{2}}\, t^{\frac{3}{2}}\, \frac{1}{3\sqrt{\pi}}$ and conclude that $(X_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are not independent (despite being uncorrelated and Gaussian processes).

Solution 10.1

(a) The function $\operatorname{sign}(\cdot)$ is a bounded function and so the stochastic integral is well defined and a continuous local martingale. Its quadratic variation is

$$\langle X\rangle_t = \int_0^t \operatorname{sign}^2(B_s)\, ds = \int_0^t ds = t,$$

and hence by Levy's characterization theorem, we see that $(X_t)_{t\geq 0}$ is a Brownian motion. For the second part, by applying the Itô isometry, we obtain that:

$$E\big[X_t\, B_s\big] = E\Big[\int_0^s dB_u \int_0^t \operatorname{sign}(B_u) dB_u\Big] = E\Big[\int_0^{\min(s,t)} \operatorname{sign}(B_s)\, ds\Big].$$

By Fubini, we get that

$$E\Big[\int_0^{\min(s,t)} \operatorname{sign}(B_s)\, ds\Big] = \int_0^{\min(s,t)} E\big[\operatorname{sign}(B_s)\big]\, ds = 0.$$

Where the last equation follows from the symmetry of the Brownian motion, i.e.

$$E\big[\operatorname{sign}(B_s)\big] = P\big[B_s \geq 0\big] - P\big[B_s < 0\big] = 0.$$

(b) By Itô's Formula, we know that $B_t^2 = 2\int_0^t B_s\, dB_s + t$. Since $E[X_t] = 0$, we conclude using Itô's isometry and Fubini that

$$\begin{aligned} E\big[B_t^2 X_t\big] &= E\Big[\Big(\int_0^t \operatorname{sign}(B_s)\, dB_s\Big)\Big(2\int_0^t B_s\, dB_s + t\Big)\Big] \\ &= 2E\Big[\int_0^t \operatorname{sign}(B_s) B_s\, ds\Big] \\ &= 2\int_0^t E\big[|B_s|\big]\, ds. \end{aligned}$$

Finally, as $B_s \sim \mathcal{N}(0, s)$, we obtain that $E[|B_s|] = \frac{\sqrt{2s}}{\sqrt{\pi}}$. Therefore,

$$E\big[B_t^2 X_t\big] = \int_0^t \frac{\sqrt{2s}}{\sqrt{\pi}}\, ds = 2^{\frac{5}{2}} t^{\frac{3}{2}} \frac{1}{3\sqrt{\pi}}.$$

Therefore, as $E[X_t] = 0$, X and B cannot be independent.

Exercise 10.2 The goal of this exercise is to prove that *"Brownian motion does not hit points whenever $d \geq 2$"*. Let $d \geq 2$, $\Omega = C([0,\infty);\mathbb{R}^d)$ and $Y = (Y_t)_{t\geq 0}$ denote the canonical process. For each $x \in \mathbb{R}^d$, let $\mathbb{P}_x$ be the unique probability measure on $(\Omega, \mathcal{Y}^0_\infty)$ under which Y is a (d-dimensional) Brownian motion started at x.

(a) Let $0 \neq x \in \mathbb{R}^d$ and $a > 0$ such that $0 < a < |x|$ and consider the stopping time

$$\tau_{a,b} := \inf\big\{t \geq 0 \,\big|\, |Y_t| \leq a \text{ or } |Y_t| \geq b\big\}.$$

For $d \geq 3$, show that $(X_t)_{t\geq 0}$ defined by $X_t := \big|Y_{\tau_{a,b}\wedge t}\big|^{2-d}$ is a bounded martingale under $\mathbb{P}_x$. Additionally, when $d = 2$ show that $X_t = \ln(|Y_{\tau_{a,b}\wedge t})$ is also a bounded martingale.

(b) Show that

$$\text{for any } 0 \neq x \in \mathbb{R}^d, \text{ we have } \mathbb{P}_x\big[Y_t \neq 0 \text{ for all } t \geq 0\big] = 1.$$

Solution 10.2

(a) When g is a C^2 function, one can define the radial function

$$f(x) := g(|x|)$$

Then, one has for every $x \neq 0$ that

$$\Delta f(x) = g''(r) + \frac{d-1}{r} g'(r), \quad \text{with } r = |x|.$$

Now, for $d \geq 3$, consider $g(x) := x^{2-d}$, which is C^2 on $(0,\infty)$ and let $f(x) := g(|x|)$. Then we get for any $x \neq 0$, as $g(r) = r^{2-d}$, that

$$\Delta f(x) = g''(r) + \frac{d-1}{r} g'(r) = 0,$$

which means that $f(x) = |x|^{2-d}$ is harmonic in $\mathbb{R}^d \setminus \{0\}$.

By applying Itô's formula, as f is harmonic, we see that $\mathbb{P}_x$-a.s., for all $t \geq 0$

$$X_t = \big|Y_{\tau_a \wedge t}\big|^{2-d} = f(Y_{\tau_{a,b}\wedge t}) = f(x) + \int_0^{\tau_{a,b}\wedge t} \nabla f(Y_s)\, dY_s,$$

which proves that $(X_t)_{t\geq 0}$ is a local martingale. Moreover, as $d \geq 3$, we have that

$$0 \leq X_t = \big|Y_{\tau_{a,b}\wedge t}\big|^{2-d} = \frac{1}{|Y_{\tau_{a,b}\wedge t}|^{d-2}} \leq \frac{1}{a^{d-2}}.$$

Thus, since it is uniformly bounded, we obtain that X is a true martingale.

For $d \geq 2$, consider $g(x) := \ln(x)$ which is C^2 on $(0,\infty)$ and let $f(x) = g(|x|)$. Then, for any $x \neq 0$

$$\Delta f(x) = 0$$

Thus by applying Itô's formula

$$X_t = f(x) + \int_0^{\tau_{a,b}\wedge t} \nabla f(Y_s)\, dY_s,$$

Thus, when $x \in (-a, b)$, X_t is a local martingale bounded by $\max\{|\ln(a)|, |\ln(b)|\}$. So it is a true martingale.

(b) We distinguish between the case $d \geq 3$ and $d = 2$. Consider first the case $d \geq 3$, where

$$\mathbb{E}_x\left[|Y_{\tau_{a,b}\wedge t}|^{2-d}\right] = |x|^{2-d}, \text{ for all } t \geq 0.$$

Now, by applying the law of iterated logarithm for each component Y^i, $i = 1, .., d$, we conclude that $\tau_{a,b}$ is $\mathbb{P}_x$-a.s. finite. Thus, we obtain by letting $t \to \infty$ and applying dominated convergence that

$$|x|^{2-d} = \lim_{t\to\infty} \mathbb{E}_x\left[|Y_{\tau_{a,b}\wedge t}|^{2-d}\right] = \mathbb{E}_x\left[|Y_{\tau_{a,b}}|^{2-d}\right] = a^{2-d}\,\mathbb{P}_x\left[|Y_{\tau_{a,b}}| = a\right] + b^{2-d}\,\mathbb{P}_x\left[|Y_{\tau_{a,b}}| = b\right].$$

Moreover, as $\mathbb{P}_x\left[|Y_{\tau_{a,b}}| = a\right] + \mathbb{P}_x\left[|Y_{\tau_{a,b}}| = b\right] = 1$, we obtain for any $0 < a < |x| < b$ that

$$\mathbb{P}_x\left[|Y_{\tau_{a,b}}| = a\right] = \frac{|x|^{2-d} - b^{2-d}}{a^{2-d} - b^{2-d}}, \quad \mathbb{P}_x\left[|Y_{\tau_{a,b}}| = b\right] = \frac{a^{2-d} - |x|^{2-d}}{a^{2-d} - b^{2-d}}. \tag{1}$$

Now, consider the stopping times

$$\tau_0 = \inf\left\{t \geq 0 \,\middle|\, |Y_t| = 0\right\}$$
$$\sigma_b := \inf\left\{t \geq 0 \,\middle|\, |Y_t| \geq b\right\}.$$

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence decreasing to 0 such that $a_n < |x|$ for all n. We deduce from (1), as $d \geq 3$, that for any fixed $b > |x|$,

$$\mathbb{P}_x[\tau_0 < \sigma_b] = \mathbb{P}_x\Big[\bigcap_n \{\tau_{a_n} < \sigma_b\}\Big] = \lim_{n\to\infty} \mathbb{P}_x\left[\tau_{a_n} < \sigma_b\right] = \lim_{n\to\infty} \mathbb{P}_x\left[|Y_{\tau_{a_n,b}}| = a_n\right] = 0. \tag{2}$$

Let $(b_n)_{n\in\mathbb{N}}$ be a sequence which increases to infinity such that $|x| < b_n$ for all n. Applying (2), we observe that

$$\mathbb{P}_x\left[Y_t = 0 \text{ for some } t \geq 0\right] = \mathbb{P}_x\Big[\bigcup_n \{\tau_0 < \sigma_{b_n}\}\Big] = \lim_{n\to\infty} \mathbb{P}_x\left[\tau_0 < \sigma_{b_n}\right] = 0.$$

Next, we consider the case when $d = 2$. Using the same argument as in the case $d \geq 3$, but with respect to $f(y) := \log \frac{1}{|y|}$, yields that for any $a < |x| < b$

$$\mathbb{P}_x\left[|Y_{\tau_{a,b}}| = a\right] = \frac{\log \frac{b}{|x|}}{\log \frac{b}{a}}, \quad \mathbb{P}_x\left[|Y_{\tau_{a,b}}| = b\right] = \frac{\log \frac{|x|}{a}}{\log \frac{b}{a}}.$$

With the same arguments as in the case $d \geq 3$, we obtain that

$$\mathbb{P}_x\left[Y_t = 0 \text{ for some } t \geq 0\right] = 0.$$

Exercise 10.3 Let B be a Brownian motion in $\mathbb{R}^3$, $0 \neq x \in \mathbb{R}^3$ and define the process $M = (M_t)_{t\geq 0}$ by

$$M_t = \frac{1}{|x + B_t|}.$$

This is well defined since one can show that $P[B_t = -x \text{ for some } t \geq 0] = 0$.

(a) Show that M is a continuous local martingale.
Hint: Use Itô's formula.

Moreover, show that M is bounded in L^2, i.e., $\sup_{t\geq 0} E[|M_t|^2] < \infty$.
Hint: For any $t \geq 0$, show that

$$E\left[|M_t|^2 1_{\{|M_t|\geq \frac{2}{|x|}\}}\right] = (2\pi t)^{-\frac{3}{2}} \int\limits_{|y|\leq \frac{|x|}{2}} \frac{1}{|y|^2} \exp\left(-\frac{|y-x|^2}{2t}\right) dy$$

and estimate the right-hand side from above using the reverse triangle inequality.

(b) Show that M is a *strict local martingale*, i.e., M is not a martingale.
Hint: Show that $E[M_t] \to 0$ as $t \to \infty$. To this end, similarly to part a), compute $E[M_t]$ and use the reverse triangle inequality as a first estimate. Then compute the resulting integral using spherical coordinates.
Remark: This is the standard example of a local martingale which is not a (true) martingale. It also shows that even good integrability properties like boundedness in L^2 are not enough to guarantee the martingale property.

Solution 10.3

(a) Since the 3-dimensional Brownian motion $B = (B^1, B^2, B^3)$ takes values in the open set $D := \mathbb{R}^d \setminus \{-x\}$ P-a.s., we can apply Itô's formula to $M_t = f(B_t)$ with $f : D \to (0, \infty)$ given by $f(y) := \frac{1}{|x+y|}$.
For $i = 1, 2, 3$, we have

$$\frac{\partial f}{\partial y^i}(y) = -\frac{x^i + y^i}{|x+y|^3}, \quad \frac{\partial^2 f}{(\partial x^i)^2}(y) = \frac{-|x+y|^2 + 3(x^i+y^i)^2}{|x+y|^5}.$$

It follows that $\Delta f = \frac{\partial^2 f}{(\partial x^1)^2} + \frac{\partial^2 f}{(\partial x^2)^2} + \frac{\partial^2 f}{(\partial x^3)^2} = 0$ on D. Hence, Itô's formula yields

$$M_t = M_0 + \int_0^t \nabla f(B_s)\, dB_s + \frac{1}{2}\int_0^t \Delta f(B_s)\, ds = \frac{1}{|x|} - \sum_{i=1}^3 \int_0^t \frac{x^i + B_s^i}{|x+B_s|^3}\, dB_s^i.$$

Thus, M is a continuous local martingale.
Let's show the second part. For $t > 0$,

$$\begin{aligned} E\left[|M_t|^2, |M_t| \geq \frac{2}{|x|}\right] &= (2\pi t)^{-\frac{3}{2}} \int\limits_{|x+y|\leq \frac{|x|}{2}} \frac{1}{|x+y|^2} \exp\left(-\frac{|y|^2}{2t}\right) dy \\ &= (2\pi t)^{-\frac{3}{2}} \int\limits_{|y|\leq \frac{|x|}{2}} \frac{1}{|y|^2} \exp\left(-\frac{|y-x|^2}{2t}\right) dy \\ &\leq (2\pi t)^{-\frac{3}{2}} \int\limits_{|y|\leq \frac{|x|}{2}} \frac{1}{|y|^2} \exp\left(-\frac{(|x|-|y|)^2}{2t}\right) dy \\ &\leq (2\pi t)^{-\frac{3}{2}} \exp\left(-\frac{|x|^2}{8t}\right) \int\limits_{|y|\leq \frac{|x|}{2}} \frac{1}{|y|^2}\, dy. \end{aligned}$$

The integral term in the preceding expression is finite since the domain of integration is 3-dimensional. Moreover, the function $t \mapsto (2\pi t)^{-\frac{3}{2}} \exp\left(-\frac{|x|^2}{8t}\right)$ is continuous on $(0,\infty)$ and converges to 0 as $t \to 0$ and $t \to \infty$, hence it is bounded on $(0,\infty)$. It follows that M is bounded in L^2.

(b) For $t > 0$, using spherical coordinates,

$$
\begin{aligned}
E[M_t] &= (2\pi t)^{-3/2} \int_{\mathbb{R}^3} \frac{1}{|x+y|} \exp\left(-\frac{|y|^2}{2t}\right) dy \\
&= (2\pi t)^{-3/2} \int_{\mathbb{R}^3} \frac{1}{|y|} \exp\left(-\frac{|y-x|^2}{2t}\right) dy \\
&\leq (2\pi t)^{-3/2} \int_{\mathbb{R}^3} \frac{1}{|y|} \exp\left(-\frac{(|y|-|x|)^2}{2t}\right) dy \\
&= (2\pi t)^{-3/2} \int_0^\infty \int_0^{2\pi} \int_0^\pi \frac{1}{r} \exp\left(-\frac{(r-|x|)^2}{2t}\right) r^2 \sin\theta \, d\theta d\varphi dr \\
&= 4\pi (2\pi t)^{-3/2} \int_0^\infty r \exp\left(-\frac{(r-|x|)^2}{2t}\right) dr \\
&= 4\pi (2\pi t)^{-3/2} \int_{-|x|}^\infty (r+|x|) \exp\left(-\frac{r^2}{2t}\right) dr \\
&= 4\pi (2\pi t)^{-3/2} \left(\int_{-|x|}^\infty r \exp\left(-\frac{r^2}{2t}\right) dr + |x| \int_{-|x|}^\infty \exp\left(-\frac{r^2}{2t}\right) dr \right) \\
&\leq 4\pi (2\pi t)^{-3/2} \left(\left[-t \exp\left(-\frac{r^2}{2t}\right) \right]_{-|x|}^\infty + |x|\sqrt{2\pi t} \right) \\
&= 4\pi (2\pi t)^{-3/2} \left(t \exp\left(-\frac{|x|^2}{2t}\right) + |x|\sqrt{2\pi t} \right) = O\left(t^{-\frac{1}{2}}\right) \quad (t \to \infty).
\end{aligned}
$$

Hence, $E[M_t] \to 0$ as $t \to \infty$. Since $E[M_0] = \frac{1}{|x|} > 0$, M cannot be a martingale.