Coordinator
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Brownian Motion and Stochastic Calculus

Exercise sheet 11

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than May 19th

Exercise 11.1 Show that for a uniformly integrable martingale M with right-continuous path.

$$\mathbb{P}\left(\sup_{t\geq 0}|M_t| \geq \epsilon\right) \leq \frac{1}{\epsilon}\mathbb{E}[|M_\infty|]$$

Exercise 11.2 Consider a probability space $(\Omega, \mathcal{F}, P)$ carrying a Brownian motion $W = (W_t)_{t\geq 0}$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the P-augmentation of the (raw) filtration generated by W. Moreover, fix $T > 0$, $a < b$, and set $F := 1_{\{a\leq W_T \leq b\}}$. The goal of this exercise is to find explicitly the integrand $H \in L^2_{\text{loc}}(W)$ in the Itô representation

$$F = E[F] + \int_0^\infty H_s \, dW_s. \tag{$\star$}$$

(a) Show that the martingale $M = (M_t)_{t\geq 0}$ given by $M_t := E[F|\mathcal{F}_t]$ has the representation

$$M_t = g(W_t, t), \quad 0 \leq t \leq T,$$

for a Borel function $g : \mathbb{R} \times [0, T) \to \mathbb{R}$. Compute g in terms of the distribution function Φ of the standard normal distribution.

(b) Apply Itô's formula to $g(W_t, t)$. Note that the process $(W_t, t)_{t\geq 0}$ restricted to $\Omega \times (0, T)$ takes values in the open set $\mathbb{R} \times (0, T)$.
Hint: Since M is a martingale, you do not need to calculate all the terms in Itô's formula.

(c) From **b)**, deduce a candidate for H and show that it works for Itô's representation of F in $(\star)$.

Exercise 11.3 The objective of this problem is to prove that Itô's representation theorem does not hold for filtrations that are not Brownian, i.e., we want to find two square integrable-martingales X, M such that under the filtration generated by M, X is a martingale but X cannot be represented as a stochastic integral with respect to M.

To do this take $B_\cdot$, $W_\cdot$ 2 independent Brownian motion, and define $M_t := \int_0^t B_s dW_s$ and $X_t := B_t^2 - t$, and take $\mathcal{F}_t^M = \sigma(M_s : s \leq t) \vee \mathcal{N}$, for $\mathcal{N}$ the family of P-nullsets.

(a) Show that X_t is $\mathcal{F}_t^M$-measurable. Furthermore, $M_\cdot$, $X_\cdot$ are $\mathcal{F}_\cdot^M$-martingales, such that $\mathbb{E}\left[M_t^2\right], \mathbb{E}\left[X_t^2\right] < \infty$.

(b) Show that it doesn't exist an adapted process H such that $\mathbb{E}\left[\int_0^t H_s^2 d\langle M\rangle_s\right] < \infty$, and a.s. $\int_0^t H_s dM_s = X_t$.
Hint: Compute $\mathbb{E}\left[\left(\int_0^t H_s dM_s - X_t\right)^2\right]$