Brownian Motion and Stochastic Calculus

Exercise sheet 11

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than May 12th

Exercise 11.1 Show that for a uniformly integrable martingale M with right-continuous path.

$$\mathbb{P}\left(\sup_{t\geq 0}|M_t| \geq \epsilon\right) \leq \frac{1}{\epsilon}\mathbb{E}[|M_\infty|]$$

Solution 11.1 Define $\tau_\epsilon := \inf\{s \in \mathbb{R} : |M_s| > \epsilon\}$ and note that $|M_{\tau_\epsilon}| \geq \epsilon$, thus

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\{\sup_{t\geq 0}|M_t|\geq\epsilon\}}\right] &\leq \frac{1}{\epsilon}\mathbb{E}\left[|M_{\tau_\epsilon}|\mathbf{1}_{\{\sup_{t\geq 0}|M_t|\geq\epsilon\}}\right] \\
&= \frac{1}{\epsilon}\mathbb{E}\left[\liminf_t |M_{t\wedge\tau_\epsilon}|\mathbf{1}_{\{\sup_{t\geq 0}|M_t|\geq\epsilon\}}\right] \\
&\leq \frac{1}{\epsilon}\liminf_t \mathbb{E}\left[|M_{t\wedge\tau_\epsilon}|\right] \\
&\overset{(2.3.8)}{\leq} \frac{1}{\epsilon}\liminf_t \mathbb{E}\left[|\mathbb{E}\left[M_\infty \mid \mathcal{F}_{\tau_\epsilon\wedge t}\right]|\right] \\
&\leq \frac{1}{\epsilon}\mathbb{E}\left[|M_\infty|\right]
\end{aligned}$$

Exercise 11.2 Consider a probability space $(\Omega, \mathcal{F}, P)$ carrying a Brownian motion $W = (W_t)_{t\geq 0}$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the P-augmentation of the (raw) filtration generated by W. Moreover, fix $T > 0$, $a < b$, and set $F := 1_{\{a\leq W_T\leq b\}}$. The goal of this exercise is to find explicitly the integrand $H \in L^2_{\mathrm{loc}}(W)$ in the Itô representation

$$F = E[F] + \int_0^\infty H_s \, dW_s. \qquad (\star)$$

(a) Show that the martingale $M = (M_t)_{t\geq 0}$ given by $M_t := E[F|\mathcal{F}_t]$ has the representation

$$M_t = g(W_t, t), \quad 0 \leq t \leq T,$$

for a Borel function $g : \mathbb{R} \times [0, T) \to \mathbb{R}$. Compute g in terms of the distribution function Φ of the standard normal distribution.

(b) Apply Itô's formula to $g(W_t, t)$. Note that the process $(W_t, t)_{t\geq 0}$ restricted to $\Omega \times (0, T)$ takes values in the open set $\mathbb{R} \times (0, T)$.
Hint: Since M is a martingale, you do not need to calculate all the terms in Itô's formula.

(c) From **b)**, deduce a candidate for H and show that it works for Itô's representation of F in $(\star)$.

Solution 11.2

(a) By the Markov property of Brownian motion, we have for any $0 \leq t < T$,

$$M_t = E[1_{\{a\leq W_T\leq b\}}|\mathcal{F}_t] = K_{T-t}(W_t, [a, b])$$

where K is the Gaussian transition kernel. Define $g : \mathbb{R} \times [0, T) \to \mathbb{R}$ by

$$g(x, t) = K_{T-t}(x, [a, b]).$$

Then, denoting the standard normal distribution function by Φ, we have

$$g(x, t) = \Phi\left(\frac{b - x}{\sqrt{T - t}}\right) - \Phi\left(\frac{a - x}{\sqrt{T - t}}\right).$$

In particular, g is $C^{2,1}$ on $\mathbb{R} \times (0, T)$.

Alternative computation. Noting that W_t is $\mathcal{F}_t$-measurable and $W_T - W_t \sim \mathcal{N}(0, T - t)$ is independent of $\mathcal{F}_t$, we can compute

$$\begin{aligned} M_t &= E[F|\mathcal{F}_t] = P[a \leq W_T \leq b|\mathcal{F}_t] = P[a - W_t \leq W_T - W_t \leq b - W_t|\mathcal{F}_t] \\ &= \Phi\left(\frac{b - W_t}{\sqrt{T - t}}\right) - \Phi\left(\frac{a - W_t}{\sqrt{T - t}}\right) = g(W_t, t). \end{aligned}$$

(b) Since $M_t = g(W_t, t)$ is a martingale, the sum of all finite variation terms in Itô's formula applied to $g(W_t, t)$ vanishes and we obtain for $t \in (0, T)$ that

$$M_t - M_0 = \int_0^t \frac{\partial g}{\partial x}(W_s, s) \, dW_s = \int_0^t \frac{1}{\sqrt{T - s}} \left(\varphi\left(\frac{a - W_s}{\sqrt{T - s}}\right) - \varphi\left(\frac{b - W_s}{\sqrt{T - s}}\right)\right) dW_s, \qquad (1)$$

where $\varphi = \Phi'$ denotes the standard normal density.

(c) Since $x\varphi(x) \to 0$ as $x \to \pm\infty$, it is easy to see that the integrand in (1) converges P-a.s. to 0 as $s \uparrow T$. Hence,

$$H := \frac{1}{\sqrt{T-s}} \left(\varphi\left(\frac{a - W_s}{\sqrt{T-s}} \right) - \varphi\left(\frac{b - W_s}{\sqrt{T-s}} \right) \right) 1_{[\![0,T[\![}$$

is a continuous, adapted process. Thus, $H \in L^2_{\text{loc}}(W)$ and $(\star)$ yields for $0 \leq t < T$ that

$$M_t = M_0 + \int_0^t H_s \, dW_s. \tag{2}$$

Since both sides in (2) are local martingales on $[0, \infty)$ and hence continuous, we can let $t \uparrow T$ to get

$$M_T = M_0 + \int_0^T H_s \, dW_s.$$

To conclude, it suffices to note that $M_T = F$, $M_0 = E[F]$, and $\int_0^T H_s \, dW_s = \int_0^\infty H_s \, dW_s$ since H is zero on $[\![T, \infty]\!]$. Moreover, $\int H \, dW$ is a martingale as M is one.

Exercise 11.3 The objective of this problem is to prove that Itô's representation theorem does not hold for filtrations that are not Brownian, i.e., we want to find two square integrable-martingales X, M such that under the filtration generated by M, X is a martingale but X cannot be represented as a stochastic integral with respect to M.

To do this take $B_\cdot$, $W_\cdot$ 2 independent Brownian motion, and define $M_t := \int_0^t B_s dW_s$ and $X_t := B_t^2 - t$, and take $\mathcal{F}_t^M = \sigma(M_s : s \leq t) \vee \mathcal{N}$, for $\mathcal{N}$ the family of P-nullsets.

(a) Show that X_t is $\mathcal{F}_t^M$-measurable. Furthermore, $M_\cdot$, $X_\cdot$ are martingales, such that $\mathbb{E}\left[M_t^2\right], \mathbb{E}\left[X_t^2\right] < \infty$.

(b) Show that it doesn't exist an adapted process H such that $\mathbb{E}\left[\int_0^t H_s^2 d\langle M\rangle_s\right] < \infty$, and a.s. $\int_0^t H_s dM_s = X_t$.
Hint: Compute $\mathbb{E}\left[\left(\int_0^t H_s dM_s - X_t\right)^2\right]$

Solution 11.3

(a) Note that there exists a deterministic sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$,such that a.s. $\langle M_t\rangle = \lim \sum_{t_i\in\Pi_n}(M_{t_{i+1}\wedge t} - M_{t_i\wedge t})^2 = \int_0^t B_t^2 dt$, in consequence X_t is $\mathcal{F}_t^M$ measurable. Additionally, $X_{t+h} - X_t$ is independent of $\mathcal{F}_t^M$, thus X_t is an $\mathcal{F}_t^M$-martingale. To finish, note that

$$\mathbb{E}\left[X_t^2\right] = \mathbb{E}\left[B_t^4\right] - 2t^2\mathbb{E}\left[B_t^2\right] + t^4 < \infty,$$
$$\mathbb{E}\left[M_t^2\right] = \int_0^t \int_0^t \mathbb{E}\left[B_s B_u\right] dx du < \infty.$$

(b) Thanks to Itô's formula a.s. $X_s = 2\int_0^t B_s dB_s$. Additionally, a.s. $\int_0^t H_s dX_s = \int_0^t H_s B_s dW_s$. If a.s. $\int_0^t H_s dM_s = X_t$,

$$0 = \mathbb{E}\left[\left(\int_0^t H_s dM_s - X_t\right)^2\right] = \mathbb{E}\left[\int_0^t H_s B_s dW_s - 2\int_0^t B_s dB_s\right] = 4\mathbb{E}\left[\int_0^t B_s^2 dt\right] + \mathbb{E}\left[\int_0^t H_s^2 B_s^2 ds\right],$$

which is a contradiction.