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Brownian Motion and Stochastic Calculus

Exercise sheet 12

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than May 26th

Exercise 12.1 The aim of this exercise is to show that

> "If you run Brownian motion in two dimensions for a positive amount of time, it will write your name [in cursive script, without dotted i's or crossed t's]."

Thinking of the function $g : [0,1] \to \mathbb{R}^2$ with $g(0) = (0,0)$, as our signature, we can make a precise statement. Take $(B_t)_{t\in[0,1]}$ a two-dimensional Brownian motion on $[0,1]$ and note that for any $[a,b] \subset [0,1]$ the process

$$X_t^{(a,b)} = \sqrt{b-a}\left(B_{a+\frac{t}{b-a}} - B_a\right)$$

is again a Brownian motion on $[0,1]$. The Brownian motion spells your name (to precision $\epsilon > 0$) on the interval (a,b) if $\mathbb{P}$-almost surely

$$\sup_{t\in[0,1]} |X_t^{(a,b)} - g(t)| < \epsilon.$$

We say that the Brownian motion writes your name if $\mathbb{P}$-almost surely

$$\sup_{t\in[0,1]} \left| X_t^{(\frac{1}{2^{n+1}},\frac{1}{2^n})} - g(t) \right| < \epsilon, \quad \text{for infinitely many } n.$$

(a) Argue why the result can be proved once we show that

$$\mathbb{P}\left(\sup_{t\in[0,1]} |B_t - g(t)| < \epsilon\right) > 0, \quad \forall \epsilon > 0. \tag{1}$$

(b) Consider an individual who does not even make an X as signature, i.e. $g(t) = (0,0)$ for all $t \in [0,1]$. Show that

$$\mathbb{P}\left(\sup_{t\in[0,1]} |B_t| < \epsilon\right) > 0, \quad \forall \epsilon > 0.$$

(c) Complete the solution of the problem using **b)** and Girsanov theorem.

Exercise 12.2 Dirichlet Problem Let D be a bounded open set of $\mathbb{R}^d$ and f a continuous function on ∂D. Suppose there exist a function $g : \bar{D} \mapsto \mathbb{R}$ continuous in ∂D and of class C^2 in D, such that $g = f$ in ∂D and $\Delta g = 0$ in D. Let $x \in D$ and $(B_t)_{t\geq 0}$ a d-dimensional Brownian motion starting from x. Define $T := \inf\{t \geq 0 : B_t \notin D\}$. Show that

$$g(x) = \mathbb{E}\left[f(B_t)\right]$$

and conclude that if such a g exists it is unique.
HINT: It may be useful to define $T_\epsilon := \inf\{s \leq t : dist(B_t, \partial D) \leq \epsilon\}$.

Exercise 12.3 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ satisfying the usual conditions.

(a) Consider the *Ornstein-Uhlenbeck process*

$$X_t = xe^{-\lambda t} + \nu(1 - e^{-\lambda t}) + \int_0^t \sigma e^{\lambda(s-t)} dW_s, \quad t \geq 0 \tag{2}$$

for an $x \in \mathbb{R}$, where ν and $\lambda, \sigma > 0$ are real constants. Show that X satisfies the Ornstein-Uhlenbeck SDE:

$$dX_t = \lambda(\nu - X_t)dt + \sigma dW_t, \quad X_0 = x.$$

Hint: Apply Itô's formula to $f(x,t) = xe^{\lambda t}$.

(b) Calculate the mean and variance functions of X:

$$T \mapsto \mathbb{E}[X_T], \quad \text{and} \quad T \mapsto \mathrm{Var}[X_T].$$

Exercise 12.4 Matlab Exercise Given a finite time horizon $T = 1$, the aim of this exercise is to simulate the Ornstein-Uhlenbeck process and the Cox-Ingersoll-Ross process from (Ex 11-2) on the time interval $[0, T]$ using the *Euler-Maruyama scheme.*[1]
To this end, let W be a dimensional Brownian motion. We define an equidistant decomposition $\{0 = t_0 < \ldots < t_n = T\}$ of the interval $[0, T]$ by setting

$$t_i := \frac{i}{M} T, \quad i = 0, \ldots, M = 10^3.$$

If X is a process on the interval $[0, T]$ satisfying the stochastic differential equation

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

with initial condition $X_0 = x$ for an $x \in \mathbb{R}$, and $t_0 = 0 < t_1 < \ldots < t_M = T$ is a given discretization of the time interval $[0, T]$, then an *Euler-Maruyama approximation*[2] of X is given by the iterative scheme: $X_0 = x$ and

$$X_{t_{i+1}} = X_{t_i} + a(t_i, X_{t_i})(t_{i+1} - t_i) + b(t_i, X_{t_i})(W_{t_{i+1}} - W_{t_i}), \quad i = 0, \ldots, M-1.$$

(a) Simulate 10 sample paths of the OU-process X from Ex 11-2 a) with $\lambda = 1$, $\nu = 1.2$, $\sigma = 0.3$ and $X_0 = 1$.

(b) Use Monte-Carlo simulation ($N = 10^5$) to compute $\mathbb{E}[X_1], \mathbb{E}[X_1^2], \mathbb{E}[X_1^+]$.

[1] This is the stochastic version of the Euler-scheme for ODEs.
[2] As a reference for the Euler-Maruyama approximation see for example Section 3.2 of *Numerical Solution of SDE Through Computer Experiments* (Kloeden, Platen, Schurz).