Brownian Motion and Stochastic Calculus

Exercise sheet 12

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than May 26th

Exercise 12.1 The aim of this exercise is to show that

> "If you run Brownian motion in two dimensions for a positive amount of time, it will write your name [in cursive script, without dotted i's or crossed t's]."

Thinking of the function $g : [0,1] \to \mathbb{R}^2$ with $g(0) = (0,0)$, as our signature, we can make a precise statement. Take $(B_t)_{t\in[0,1]}$ a two-dimensional Brownian motion on $[0,1]$ and note that for any $[a,b] \subset [0,1]$ the process

$$X_t^{(a,b)} = \sqrt{b-a}\left(B_{a+\frac{t}{b-a}} - B_a\right)$$

is again a Brownian motion on $[0,1]$. The Brownian motion spells your name (to precision $\epsilon > 0$) on the interval (a,b) if $\mathbb{P}$-almost surely

$$\sup_{t\in[0,1]} |X_t^{(a,b)} - g(t)| < \epsilon.$$

We say that the Brownian motion writes your name if $\mathbb{P}$-almost surely

$$\sup_{t\in[0,1]} \left|X_t^{(\frac{1}{2^{n+1}},\frac{1}{2^n})} - g(t)\right| < \epsilon, \quad \text{for infinitely many } n.$$

(a) Argue why the result can be proved once we show that

$$\mathbb{P}\left(\sup_{t\in[0,1]} |B_t - g(t)| < \epsilon\right) > 0, \quad \forall \epsilon > 0. \tag{1}$$

(b) Consider an individual who does not even make an X as signature, i.e. $g(t) = (0,0)$ for all $t \in [0,1]$. Show that

$$\mathbb{P}\left(\sup_{t\in[0,1]} |B_t| < \epsilon\right) > 0, \quad \forall \epsilon > 0.$$

(c) Complete the solution of the problem using **b)** and Girsanov theorem.

Solution 12.1

(a) First note that

$$\mathbb{P}\left(\sup_{t\in[0,1]} |B_t - g(t)| < \epsilon\right) > 0, \quad \forall \epsilon > 0$$
$$\Rightarrow \sum_{n\in\mathbb{N}} \mathbb{P}\left(\sup_{t\in[0,1]} \left|X_t^{(\frac{1}{2^{n+1}},\frac{1}{2^n})} - g(t)\right| < \epsilon\right) = \infty$$

Due to the fact that $(X_t^{(\frac{1}{2^{n+1}},\frac{1}{2^n})})_{n\in\mathbb{N}}$ is an iid sequence. We can use Borel-Cantelli 2 to conclude.

(b) Note that

$$\mathbb{P}\left(\sup_{t\leq 1}|B_t| < \epsilon\right) \geq \mathbb{P}\left(\sup_{t\leq 1}|B_t^{(1)}| < \epsilon/2\right)\mathbb{P}\left(\sup_{t\leq 1}|B_t^{(2)}| < \epsilon/2\right) > 0.$$

(c) We can always assume that $g \in C^1$, due to the fact that this is dense in C_0^∞. Now, define $\mathbb{P}$ the probability measure where B is a Brownian motion. Define $\tilde{\mathbb{P}}$ such that

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(\int_0^1 g'(t)dB_t - \frac{1}{2}\int_0^1 (g')^2(t)dt\right) > 0,$$

under $\tilde{P}$, $\tilde{B}_t = B_t - g(t)$ is a Brownian motion. Thus due to the fact that $\mathbb{P} \simeq \tilde{\mathbb{P}}$

$$\tilde{\mathbb{P}}\left(\sup_{t\leq 1}|B_t - g(t)| < \epsilon\right) > 0 \Rightarrow \mathbb{P}\left(\sup_{t\leq 1}|B_t - g(t)| < \epsilon\right) > 0.$$

From where we conclude.

Exercise 12.2 Dirichlet Problem Let D be a bounded open set of $\mathbb{R}^d$ and f a continuous function on ∂D. Suppose there exist a function $g : \bar{D} \mapsto \mathbb{R}$ continuous in ∂D and of class C^2 in D, such that $g = f$ in ∂D and $\Delta g = 0$ in D. Let $x \in D$ and $(B_t)_{t\geq 0}$ a d-dimensional Brownian motion starting from x. Define $T := \inf\{t \geq 0 : B_t \notin D\}$. Show that

$$g(x) = \mathbb{E}\left[f(B_t)\right]$$

and conclude that if such a g exists it is unique.
Hint: It may be useful to define $T_\epsilon := \inf\{s \leq t : dist(B_t, \partial D) \leq \epsilon\}$.

Solution 12.2 Note that B^{T_ϵ} is a.s. in D. Thus we can apply Itô's formula to $g(B^{T_\epsilon})$ to get

$$g(B_t^{T_\epsilon}) = g(x) + \int_0^{t\wedge T_\epsilon} \nabla g(B_s) dB_s,$$

thus $(g(B_t^{T_\epsilon}))_{t\geq 0}$ is a bounded local martingale. Thus, it is a martingale

$$g(x) = \mathbb{E}\left[g(B_t^{T_\epsilon})\right] \to \mathbb{E}\left[f(B_T)\right], \quad \text{as } \epsilon \to 0.$$

where in the last step we used bounded convergence and the fact that g is continuous.

The fact that g is a function of f implies that if g exists it is unique.

Exercise 12.3 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ satisfying the usual conditions.

(a) Consider the *Ornstein-Uhlenbeck process*

$$X_t = xe^{-\lambda t} + \nu(1 - e^{-\lambda t}) + \int_0^t \sigma e^{\lambda(s-t)} dW_s, \quad t \geq 0 \tag{2}$$

for an $x \in \mathbb{R}$, where ν and $\lambda, \sigma > 0$ are real constants. Show that X satisfies the Ornstein-Uhlenbeck SDE:

$$dX_t = \lambda(\nu - X_t)dt + \sigma dW_t, \quad X_0 = x.$$

Hint: Apply Itô's formula to $f(x,t) = xe^{\lambda t}$.

(b) Calculate the mean and variance functions of X:

$$T \mapsto \mathbb{E}[X_T], \quad \text{and} \quad T \mapsto \text{Var}[X_T].$$

Solution 12.3

(a) Consider the function $f(x,t) = xe^{\lambda t}$. Itô's formula applied to f yields

$$\begin{aligned}
f(X_t, t) &= X_0 e^{\lambda 0} + \int_0^t X_t \lambda e^{\lambda s} ds + \int_0^t e^{\lambda s} dX_s \\
&= X_0 e^{\lambda 0} + \int_0^t X_t \lambda e^{\lambda s} ds + \int_0^t e^{\lambda s} \lambda(\nu - X_s) ds + \int_0^t e^{\lambda s} \sigma dW_s \\
&= X_0 e^{\lambda 0} + \int_0^t e^{\lambda s} \lambda \nu ds + \int_0^t e^{\lambda s} \sigma dW_s \\
&= X_0 e^{\lambda 0} + \nu(e^{\lambda t} - 1) + \int_0^t e^{\lambda s} \sigma dW_s.
\end{aligned}$$

Now multiplying both sides by $e^{-\lambda t}$ and inserting the initial value $X_0 = x$ $\mathbb{P}$-a.s. gives

$$f(X_t, t)e^{-\lambda t} = X_t = xe^{-\lambda t} + \nu(1 - e^{-\lambda t}) + \int_0^t e^{\lambda(s-t)} \sigma dW_s, \quad t \geq 0.$$

(b) To compute $\mathbb{E}[X_t]$ we first show that $(\int_0^t \sigma e^{\lambda(s-t)} dW_s)_{0\leq t\leq T}$ is a $(\mathbb{P}, \mathcal{F})$-martingale on $[0, T]$ with mean 0. We notice that the integrand is continuous and adapted (and hence predictable and locally bounded). Therefore, the stochastic integral is a local-martingale. Since

$$\mathbb{E}\left[\int_0^T \sigma^2 e^{2\lambda(s-T)} ds\right] = \frac{\sigma^2}{2\lambda}\left(1 - e^{-2\lambda T}\right) < \infty,$$

it is even a true martingale by EX 7-2. Thus,

$$\mathbb{E}[X_T] = xe^{-\lambda T} + \nu(1 - e^{-\lambda T}).$$

Moreover, using Itô's isometry we have

$$\begin{aligned}
Var[X_T] &= \mathbb{E}[(X_T - \mathbb{E}[X_T])^2] \\
&= \mathbb{E}\left[\left(\sigma \int_0^T e^{\lambda(s-T)} dW_s\right)^2\right] \\
&= \sigma^2 \mathbb{E}\left[\int_0^T e^{2\lambda(s-T)} ds\right] \\
&= \frac{\sigma^2}{2\lambda}\left(1 - e^{-2\lambda T}\right).
\end{aligned}$$

Exercise 12.4 Matlab Exercise Given a finite time horizon $T = 1$, the aim of this exercise is to simulate the Ornstein-Uhlenbeck process and the Cox-Ingersoll-Ross process from (Ex 11-2) on the time interval $[0, T]$ using the *Euler-Maruyama scheme.*[1]
To this end, let W be a dimensional Brownian motion. We define an equidistant decomposition $\{0 = t_0 < \ldots < t_n = T\}$ of the interval $[0, T]$ by setting

$$t_i := \frac{i}{M}T, \quad i = 0, \ldots, M = 10^3.$$

If X is a process on the interval $[0, T]$ satisfying the stochastic differential equation

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

with initial condition $X_0 = x$ for an $x \in \mathbb{R}$, and $t_0 = 0 < t_1 < \ldots < t_M = T$ is a given discretization of the time interval $[0, T]$, then an *Euler-Maruyama approximation*[2] of X is given by the iterative scheme: $X_0 = x$ and

$$X_{t_{i+1}} = X_{t_i} + a(t_i, X_{t_i})(t_{i+1} - t_i) + b(t_i, X_{t_i})(W_{t_{i+1}} - W_{t_i}), \quad i = 0, \ldots, M-1.$$

(a) Simulate 10 sample paths of the OU-process X from Ex 11-2 a) with $\lambda = 1$, $\nu = 1.2$, $\sigma = 0.3$ and $X_0 = 1$.

(b) Use Monte-Carlo simulation ($N = 10^5$) to compute $\mathbb{E}[X_1], \mathbb{E}[X_1^2], \mathbb{E}[X_1^+]$.

Solution 12.4

```
function bmsc114a
% In this exercise we simulated N paths of a OU process
% dX_t = \lambda (\nu-X_t) dt + \sigma dW_t, X_0 =x and simulate the
% expecation of X_1, X_1^2, X_1^+
tic
%% parameter input
% horizon
T=1;
% sample size
Nsimu=10^5;
Nplot=Nsimu;
% grid points
M=10^3;
% volatility
sigma=0.3;
lambda=1;
nu=1.2;
x=1;
% time step
dt= T/M;

% theoretical value for the expectation, second moment and pos part
mutilde = x*exp(-lambda*T)+ nu*(1-exp(-lambda*T));
sigmatilde= sqrt(sigma^2/(2*lambda)*(1-exp(-2*lambda*T)));
theoreticalvalueexp= mutilde;
theoreticalvaluesec= sigmatilde^2+mutilde^2;
```

[1] This is the stochastic version of the Euler-scheme for ODEs.
[2] As a reference for the Euler-Maruyama approximation see for example Section 3.2 of *Numerical Solution of SDE Through Computer Experiments* (Kloeden, Platen, Schurz).

```
theoreticalvaluepos= mutilde*normcdf(mutilde/sigmatilde)+sigmatilde/sqrt(2*pi)*exp(-1/2*(mutilde/sigmatilde)^2);
%% Simulation
% BM
BM = [zeros(1,Nplot);sqrt(T/M)*cumsum(randn(M,Nplot))];
OU = [x*ones(1,Nplot);zeros(M,Nplot)];
% the process X
for i =1:M
    OU(i+1,:)=OU(i,:)+lambda*(nu-OU(i,:))*dt+ sigma.*(BM(i+1,:)-BM(i,:));
end

%plot the first 10 sample paths
timegrid= 0:dt:T;
plot(timegrid,OU(:,1:10))

%compute simulated value
simulatedvalueexp= mean(OU(end,:));
simulatedvaluesec= mean(OU(end,:).^2);
simulatedvaluepos= mean(subplus(OU(end,:)));

disp('Exact values: Expectation/2.Moment/pos. part')
disp([theoreticalvalueexp;theoreticalvaluesec;theoreticalvaluepos])
disp('Estimated value: Expectation/2.Moment/pos. part')
disp([simulatedvalueexp;simulatedvaluesec;simulatedvaluepos])

%estimated variance
%estvarexp= var(OU(end,:));
%estvarsec= var(OU(end,:).^2);
%estvarpos= var(subplus(OU(end,:)));
% confidence interval using CLT
%cfplusexp=simulatedvalueexp+1.96*sqrt(estvarexp/Nsimu);
%cfminusexp=simulatedvalueexp-1.96*sqrt(estvarexp/Nsimu);
%cfplussec=simulatedvaluesec+1.96*sqrt(estvarsec/Nsimu);
%cfminussec=simulatedvaluesec-1.96*sqrt(estvarsec/Nsimu);
%cfpluspos=simulatedvaluepos+1.96*sqrt(estvarpos/Nsimu);
%cfminuspos=simulatedvaluepos-1.96*sqrt(estvarpos/Nsimu);

%disp('Confidence interval: ')
%disp([cfminusexp,cfplusexp; cfminussec,cfplussec;cfminuspos,cfpluspos])

toc
```

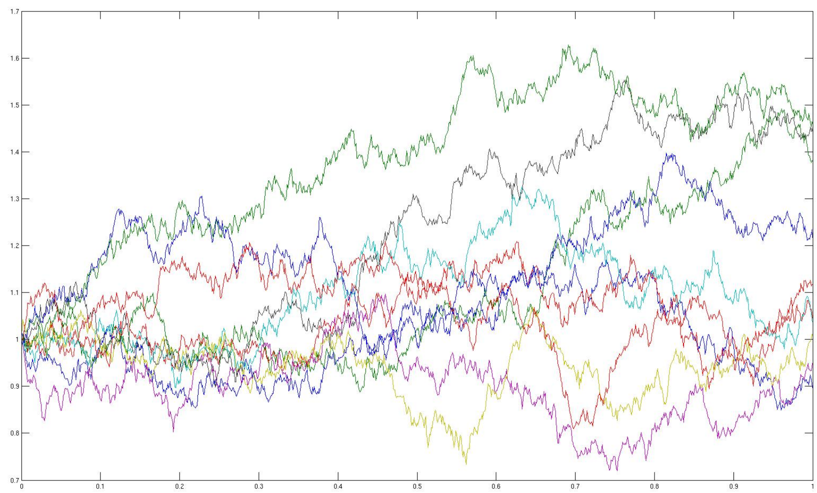

Figure 1: 10 sample paths of a OU process