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# Brownian Motion and Stochastic Calculus

## Exercise sheet 13

*Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than June 2nd*

**Exercise 13.1** Let $(B_t)_{t\geq 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. Consider the SDE

$$X_t = \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dB_s, \quad X_0 = 0 \tag{1}$$

where $b(x) := 3x^{1/3}$ and $\sigma(x) := 3x^{2/3}$. Show that the SDE has uncountably many strong solutions of the form

$$X_t^{(\Theta)} = \begin{cases} 0, & 0 \leq t < \beta_\Theta, \\ B_t^3, & \beta_\Theta \leq t < \infty, \end{cases}$$

where $0 \leq \Theta \leq \infty$ is any fixed constant and $\beta_\Theta := \inf\{s \geq \Theta \,|\, B_s = 0\}$.

**Exercise 13.2** Recall that $C_0(\mathbb{R})$ denotes the space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ that vanish at infinity. We call $C_0^2(\mathbb{R})$ the space of twice continuously differentiable functions $f$ on $\mathbb{R}$ such that $f, f'$ and $f''$ all belong to $C_0(\mathbb{R})$. For $a, b : \mathbb{R} \to \mathbb{R}$ Lipschitz-continuous and bounded, we define the partial differential operator $\mathcal{A} : C_0^2(\mathbb{R}) \to C_0(\mathbb{R})$ by

$$\mathcal{A}f(x) = a(x)\frac{\partial f}{\partial x}(x) + \frac{1}{2}b^2(x)\frac{\partial^2 f}{\partial x^2}(x), \quad x \in \mathbb{R}.$$

The goal of this exercise is to link weak solutions of the SDE

$$dX_t = a(X_t)\, dt + b(X_t)\, dW_t \tag{$\diamond$}$$

to solutions of the martingale problem for $(\mathcal{A}, C_K^2(\mathbb{R}))$, where $C_K^2(\mathbb{R})$ denotes the subspace of compactly supported functions in $C_0^2(\mathbb{R})$.

(a) Let $X$ be a weak solution to $(\diamond)$ with initial distribution $\delta_{\{x\}}$. Show that $X$ is a solution to the martingale problem for $(\mathcal{A}, C_K^2(\mathbb{R}))$. In other words, show that if $(\Omega, \mathcal{F}, \mathbb{F}, Q, W, X)$ is a weak solution to $(\diamond)$ with initial distribution $\delta_{\{x\}}$, then the process $M^f$ defined by $M_t^f := f(X_t) - \int_0^t \mathcal{A}f(X_s)\, ds$, $t \geq 0$, is a $(Q, \mathbb{F}^X)$-martingale for each $f \in C_K^2(\mathbb{R})$.

(b) Fix $x \in \mathbb{R}$ and let $X$, defined on some probability space $(\Omega, \mathcal{F}, P)$, be a continuous process which is a solution to the martingale problem for $(\mathcal{A}, C_K^2(\mathbb{R}))$ with $X_0 = x$. Show that the process $M := X - \int_0^\cdot a(X_s)\, ds$ is a continuous local martingale with $\langle M \rangle = \int_0^\cdot b^2(X_s)\, ds$.
**Hint:** For the first assertion, use the fact that $X$ is a solution to the martingale problem for $(\mathcal{A}, C_K^2(\mathbb{R}))$ for a sequence of functions in $C_K^2(\mathbb{R})$ that approximates the identity on $\mathbb{R}$, and construct a compatible localising sequence $(\tau_n)_{n\in\mathbb{N}}$ for $M$.
For the second assertion, first follow the same strategy for the function $y \mapsto y^2, y \in \mathbb{R}$. Then show that $M^2 - \int_0^\cdot b^2(X_s)\, ds$ is a continuous local martingale by expressing it as the sum of continuous local martingales.

(c) In the setting of b), assume that $b(x) \neq 0$ for all $x \in \mathbb{R}$ and construct from $X$ a weak solution of $(\diamond)$ with initial distribution $\delta_{\{x\}}$.
**Hint:** Consider $B := \int_0^\cdot \frac{1}{b(X_s)}\, dM_s$.

**Exercise 13.3** *Linear SDEs*

Let $(B_t)_{t\in[0,T]}$ be a Brownian motion in $[0,T]$ and $a_1, a_2, b_1, b_2$ deterministic functions of time. The general form of a scalar *linear stochastic differential equation* is

$$dX_t = (a_1(t)X_t + a_2(t))\,dt + (b_1(t)X_t + b_2(t))\,dB_t. \tag{2}$$

If the coefficients are measurable and bounded on $[0,T]$, we can apply Theorem (10.14) to get existence and uniqueness of a strong solution $(X_t)_{t\in[0,T]}$ for each initial condition $x$.

(a) When $a_2(t) \equiv 0$ and $b_2(t) \equiv 0$, (2) reduces to the *homogeneous linear SDE*

$$dX_t = a_1(t)X_t dt + b_1(t)X_t dB_t. \tag{3}$$

Show that the solution of (3) with initial data $x = 1$ is given by

$$X_t = \exp\left(\int_0^t (a_1(s) - \frac{1}{2}b_1^2(s))ds + \int_0^t b_1(s)dB_s\right).$$

*Remark:* We can write (3) as

$$dX_t = X_t dY_t, \quad \text{where} \quad dY_t = a_1(t)dt + b_1(t)dB_t.$$

Analogously as in the martingale case, $(X_t)_{t\geq 0}$ is called *stochastic exponential* of $(Y)_{t\geq 0}$ and is denoted by $\mathcal{E}(Y)_t$.

(b) Find a solution of the SDE (2) with initial condition $X_0 = x$.

*Hint:* Look for a solution of the form

$$X_t = U_t V_t,$$

where

$$dU_t = a_1(t)U_t dt + b_1(t)U_t dB_t, \quad U_0 = 1$$

and

$$dV_t = \alpha(t)dt + \beta(t)dB_t, \quad V_0 = x,$$

with $\alpha(t)$ and $\beta(t)$ coefficients to be determined.

(c) Solve the *Langevin's SDE*

$$dX_t = a(t)X_t dt + dB_t, \quad X_0 = x.$$