Brownian Motion and Stochastic Calculus

Exercise sheet 13

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than June 2nd

Exercise 13.1 Let $(B_t)_{t\geq 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. Consider the SDE

$$X_t = \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dB_s, \quad X_0 = 0 \tag{1}$$

where $b(x) := 3x^{1/3}$ and $\sigma(x) := 3x^{2/3}$. Show that the SDE has uncountably many strong solutions of the form

$$X_t^{(\Theta)} = \begin{cases} 0, & 0 \leq t < \beta_\Theta, \\ B_t^3, & \beta_\Theta \leq t < \infty, \end{cases}$$

where $0 \leq \Theta \leq \infty$ is any fixed constant and $\beta_\Theta := \inf\left\{ s \geq \Theta \,\middle|\, B_s = 0 \right\}$.

Solution 13.1 We consider for any fixed $0 \leq \Theta \leq \infty$ the process $(S_t^{(\Theta)})_{t\geq 0}$

$$S_t^{(\Theta)} = \begin{cases} 0, & 0 \leq t < \beta_\Theta, \\ B_t, & \beta_\Theta \leq t < \infty \end{cases}$$

where $\beta_\Theta := \inf\left\{ s \geq \Theta \,\middle|\, B_s = 0 \right\}$. We observe that β_Θ is a stopping time (w.r.t. any filtration satisfying the usual conditions such that B is adapted to). As a consequence of Exercise 4-2, we obtain that for any semimartingale $(S_t)_{t\geq 0}$ and any stopping time τ the stopped process $(S_t^\tau)_{t\geq 0}$ is also a semimartingale. As obviously the difference of two semimartingales is again a semimartingale, we see, as $B_{\beta_\Theta} = 0$, that

$$S^{(\Theta)} = B - B^{\beta_\Theta}$$

is a continuous semimartingale. Moreover, we have for any $t \geq 0$ that $X_t^{(\Theta)} = f(S_t^{(\Theta)})$ for the C^2 function $f(x) := x^3$. Thus, by applying Itô's formula, we get that

$$\begin{aligned}
X_t^{(\Theta)} = f\left(S_t^{(\Theta)}\right) &= \left(S_0^{(\Theta)}\right)^3 + \int_0^t 3\left(S_s^{(\Theta)}\right)^2 dS_s^{(\Theta)} + 3\int_0^t S_s^{(\Theta)}\, d\langle S^{(\Theta)}\rangle_s \\
&= \int_0^t 3\left(X_s^{(\Theta)}\right)^{2/3} dS_s^{(\Theta)} + 3\int_0^t \left(X_s^{(\Theta)}\right)^{1/3} d\left\langle S^{(\Theta)}\right\rangle_s \\
&= \int_0^t 3\left(X_s^{(\Theta)}\right)^{2/3} \mathbf{1}_{s>\beta_\Theta}\, dB_s + 3\int_0^t \left(X_s^{(\Theta)}\right)^{1/3} \mathbf{1}_{s>\beta_\Theta}\, ds \\
&= \int_0^t 3\left(X_s^{(\Theta)}\right)^{2/3} dB_s + 3\int_0^t \left(X_s^{(\Theta)}\right)^{1/3} ds
\end{aligned}$$

We conclude that $X^{(\Theta)}$ solves the SDE for each $0 \leq \Theta \leq \infty$.

Exercise 13.2 Recall that $C_0(\mathbb{R})$ denotes the space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ that vanish at infinity. We call $C_0^2(\mathbb{R})$ the space of twice continuously differentiable functions f on $\mathbb{R}$ such that f, f' and f'' all belong to $C_0(\mathbb{R})$. For $a, b : \mathbb{R} \to \mathbb{R}$ Lipschitz-continuous and bounded, we define the partial differential operator $\mathcal{A} : C_0^2(\mathbb{R}) \to C_0(\mathbb{R})$ by

$$\mathcal{A}f(x) = a(x)\frac{\partial f}{\partial x}(x) + \frac{1}{2}b^2(x)\frac{\partial^2 f}{\partial x^2}(x), \quad x \in \mathbb{R}.$$

The goal of this exercise is to link weak solutions of the SDE

$$dX_t = a(X_t)\,dt + b(X_t)\,dW_t \tag{$\diamond$}$$

to solutions of the martingale problem for $(\mathcal{A}, C_K^2(\mathbb{R}))$, where $C_K^2(\mathbb{R})$ denotes the subspace of compactly supported functions in $C_0^2(\mathbb{R})$.

(a) Let X be a weak solution to ($\diamond$) with initial distribution $\delta_{\{x\}}$. Show that X is a solution to the martingale problem for $(\mathcal{A}, C_K^2(\mathbb{R}))$. In other words, show that if $(\Omega, \mathcal{F}, \mathbb{F}, Q, W, X)$ is a weak solution to ($\diamond$) with initial distribution $\delta_{\{x\}}$, then the process M^f defined by $M_t^f := f(X_t) - \int_0^t \mathcal{A}f(X_s)\,ds$, $t \geq 0$, is a $(Q, \mathbb{F}^X)$-martingale for each $f \in C_K^2(\mathbb{R})$.

(b) Fix $x \in \mathbb{R}$ and let X, defined on some probability space $(\Omega, \mathcal{F}, P)$, be a continuous process which is a solution to the martingale problem for $(\mathcal{A}, C_K^2(\mathbb{R}))$ with $X_0 = x$. Show that the process $M := X - \int_0^\cdot a(X_s)\,ds$ is a continuous local martingale with $\langle M \rangle = \int_0^\cdot b^2(X_s)\,ds$.
Hint: For the first assertion, use the fact that X is a solution to the martingale problem for $(\mathcal{A}, C_K^2(\mathbb{R}))$ for a sequence of functions in $C_K^2(\mathbb{R})$ that approximates the identity on $\mathbb{R}$, and construct a compatible localising sequence $(\tau_n)_{n\in\mathbb{N}}$ for M.
For the second assertion, first follow the same strategy for the function $y \mapsto y^2, y \in \mathbb{R}$. Then show that $M^2 - \int_0^\cdot b^2(X_s)\,ds$ is a continuous local martingale by expressing it as the sum of continuous local martingales.

(c) In the setting of b), assume that $b(x) \neq 0$ for all $x \in \mathbb{R}$ and construct from X a weak solution of ($\diamond$) with initial distribution $\delta_{\{x\}}$.
Hint: Consider $B := \int_0^\cdot \frac{1}{b(X_s)}\,dM_s$.

Solution 13.2

(a) Let $(\Omega, \mathcal{F}, \mathbb{F}, Q, W, X)$ be a weak solution of the SDE

$$dX_t = a(X_t)\,dt + b(X_t)\,dW_t$$

with initial distribution $\delta_{\{x\}}$. Fix $f \in C_K^2(\mathbb{R})$. By Itô's formula,

$$M_t^f = f(X_t) - \int_0^t \mathcal{A}f(X_s)\,ds = f(x) + \int_0^t \left(b\frac{\partial f}{\partial x}\right)(X_s)\,dW_s.$$

Note that the function $b\frac{\partial f}{\partial x}$ is continuous and compactly supported. Hence, the integrand $\left(b\frac{\partial f}{\partial x}\right)(X_s)$ is uniformly bounded. Thus, for any $T > 0$, we obtain that

$$E\left[\left\langle \int \left(b\frac{\partial f}{\partial x}\right)(X_s)\,dW_s \right\rangle_T^{1/2}\right] \leq CT^{1/2}$$

for some constant $C > 0$. Thus, by the Burkholder-Davis-Gundy inequality, we conclude that $\sup_{0\leq t\leq T}\left|\int_0^t \left(b\frac{\partial f}{\partial x}\right)(X_s)\,dW_s\right|$ is integrable (alternatively, one could have used also Lemma 4.1.18 in the script). We conclude that for any $T > 0$, the stopped process $\left(\int \left(b\frac{\partial f}{\partial x}\right)(X_s)\,dW_s\right)^T$ is a martingale and thus $\int \left(b\frac{\partial f}{\partial x}\right)(X_s)\,dW_s$ is a martingale.

(b) Fix $n \in \mathbb{N}$ and let $\tau_n := \{t \geq 0 : |X_t - x| \geq n\}$ and $f_n \in C_K^2(\mathbb{R})$ such that $f_n(y) = y$ for $y \in [x-2n, x+2n]$. Since X is a continuous solution to the martingale problem for $(\mathcal{A}, C_K^2(\mathbb{R}))$,

$$M_t^{f_n} := f_n(X_t) - \int_0^t \mathcal{A} f_n(X_s)\, ds$$

is a martingale. Note that $f_n(y) = y$ and $\mathcal{A} f_n(y) = a(y)$ for $y \in [x-n, x+n]$, and X only takes values in $[x-n, x+n]$ on $[\![0, \tau_n]\!]$ by construction. Hence, also

$$(M^{f_n})_t^{\tau_n} = f_n(X_t^{\tau_n}) - \int_0^{t \wedge \tau_n} \mathcal{A} f_n(X_s)\, ds = X_t^{\tau_n} - \int_0^{t \wedge \tau_n} a(X_s)\, ds = M_t^{\tau_n}$$

is a martingale. Since $\tau_n \uparrow \infty$ P-a.s., we conclude that M is a continuous local martingale. Choosing $g_n \in C_K^2(\mathbb{R})$ such that $g_n(y) = y^2$ for $y \in [x-2n, x+2n]$ and using the same technique as above, we obtain that

$$H_t := (X_t)^2 - \int_0^t \left(2a(X_s)X_s + b^2(X_s)\right)\, ds$$

is a continuous local martingale. Setting $A_t := \int_0^t a(X_s)\, ds$ and using integration by parts in the fourth equality,

$$\begin{aligned}
(M_t)^2 - \int_0^t b^2(X_s)\, ds &= (X_t - A_t)^2 - \int_0^t b^2(X_s)\, ds - \left((X_t)^2 - \int_0^t \left(2a(X_s)X_s + b^2(X_s)\right)\, ds\right) + H_t \\
&= -2X_t A_t + A_t^2 + 2\int_0^t a(X_s) X_s\, ds + H_t \\
&= 2\left(-X_t A_t + \int_0^t A_s\, dA_s + \int_0^t X_s\, dA_s\right) + H_t \\
&= 2\left(\int_0^t A_s\, dA_s - \int_0^t A_s\, dX_s\right) + H_t \\
&= -2\int_0^t A_s\, dM_s + H_t
\end{aligned}$$

is a continuous local martingale. Hence, $\langle M \rangle = \int_0^{\cdot} b^2(X_s)\, ds$.

(c) With M defined as in part b), let

$$B_t := \int_0^t \frac{1}{b(X_s)}\, dM_s, \quad t \geq 0.$$

Then

$$\int_0^t b(X_s)\, dB_s = M_t - M_0 = X_t - X_0 - \int_0^t a(X_s)\, ds.$$

It only remains to show that B is a Brownian motion under P. Indeed, $B \in \mathcal{M}_{0,\mathrm{loc}}^c(P)$ since $M \in \mathcal{M}_{0,\mathrm{loc}}^c(P)$ by part **b)**, so by Lévy's characterization theorem it suffices to note that

$$\langle B \rangle_t = \int_0^t \frac{1}{b^2(X_s)}\, d\langle M \rangle_s \overset{\text{b)}}{=} \int_0^t \frac{1}{b^2(X_s)}\, b^2(X_s)\, ds = t.$$

Exercise 13.3 *Linear SDEs*

Let $(B_t)_{t\in[0,T]}$ be a Brownian motion in $[0,T]$ and a_1, a_2, b_1, b_2 deterministic functions of time. The general form of a scalar *linear stochastic differential equation* is

$$dX_t = (a_1(t)X_t + a_2(t))\,dt + (b_1(t)X_t + b_2(t))\,dB_t. \tag{2}$$

If the coefficients are measurable and bounded on $[0,T]$, we can apply Theorem (10.14) to get existence and uniqueness of a strong solution $(X_t)_{t\in[0,T]}$ for each initial condition x.

(a) When $a_2(t) \equiv 0$ and $b_2(t) \equiv 0$, (2) reduces to the *homogeneous linear SDE*

$$dX_t = a_1(t)X_t dt + b_1(t)X_t dB_t. \tag{3}$$

Show that the solution of (3) with initial data $x = 1$ is given by

$$X_t = \exp\left(\int_0^t (a_1(s) - \frac{1}{2}b_1^2(s))ds + \int_0^t b_1(s)dB_s\right).$$

Remark: We can write (3) as

$$dX_t = X_t dY_t, \quad \text{where} \quad dY_t = a_1(t)dt + b_1(t)dB_t.$$

Analogously as in the martingale case, $(X_t)_{t\ge0}$ is called *stochastic exponential* of $(Y)_{t\ge0}$ and is denoted by $\mathcal{E}(Y)_t$.

(b) Find a solution of the SDE (2) with initial condition $X_0 = x$.

Hint: Look for a solution of the form

$$X_t = U_t V_t,$$

where

$$dU_t = a_1(t)U_t dt + b_1(t)U_t dB_t, \quad U_0 = 1$$

and

$$dV_t = \alpha(t)dt + \beta(t)dB_t, \quad V_0 = x,$$

with $\alpha(t)$ and $\beta(t)$ coefficients to be determined.

(c) Solve the *Langevin's SDE*

$$dX_t = a(t)X_t dt + dB_t, \quad X_0 = x.$$

Solution 13.3

(a) Write $X_t = e^{V_t}$ with $V_t = \int_0^t (a_1(s) - \frac{1}{2}b_1^2(s))ds + \int_0^t b_1(s)dB_s$. Then

$$dX_t = e^{V_t}dV_t + \frac{1}{2}e^{V_t}d\langle V\rangle_t.$$

Plug the expression for V_t :

$$\begin{aligned} dX_t &= e^{V_t}\left((a_1(t) - \frac{1}{2}b_1^2(t))dt + b_1(t)dB_t\right) + \frac{1}{2}e^{V_t}b_1^2(t)dt \\ &= X_t\left((a_1(t) + b_1(t)dB_t\right). \end{aligned}$$

(b) Let us start as in the hint. The process $(U_t)_{t\geq 0}$ is the solution of an homogeneous linear SDE and, using **a)** we know that it is given in explicit form by

$$U_t = \exp\left(\int_0^t (a_1(s) - \frac{1}{2}b_1^2(s))ds + \int_0^t b_1(s)dB_s\right).$$

Now we want to find the coefficients $a_2(t)$ and $b_2(t)$ such that $X_t = U_t V_t$. Applying product formula

$$\beta(t)U_t = b_2(t), \text{ and } \alpha(t)U_t = \alpha(t) - b_1(t)b_2(t).$$

To sum up

$$X_t = U_t\left(x + \int_0^t \frac{a_2(s) - b_1(s)b_2(s)}{U_s}ds + \int_0^t \frac{b_2(s)}{U_s}dB_s\right).$$

(c) Applying point **b)** with $U_t = \exp\left(\int_0^t a(s)ds\right)$ we find

$$X_t = \exp\left(\int_0^t a(s)ds\right)\left(X_0 + \int_0^t \exp\left(-\int_0^u a(s)ds\right)dB_u\right).$$