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# Brownian Motion and Stochastic Calculus

## Exercise sheet 14

*Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than June 9th*

**Exercise 14.1**

(a) *Change of time in SDEs* Let $(f_t)_{t\geq 0}$ be an adapted, positive, increasing, differentiable process starting from zero and consider the following SDE

$$dX_t = \sqrt{f_t'}dB_t. \tag{1}$$

Show that the process $B_{f_t}$ is a weak solution of (1).

*Remark:* In other words, given a Brownian motion $(B_t)_{t\geq 0}$ and a function $f$ satisfying the previous assumptions, there exist a Brownian motion $(\widehat{B})_{t\geq 0}$, such that

$$d\widehat{B}_{f_t} = \sqrt{f_t'}dB_t.$$

(b) Recall from **Exercise 11-3** that a solution of the SDEs

$$dX_t = -\gamma X_t dt + \sigma dB_t, \quad X_0 = x, \tag{2}$$

is called Ornstein-Uhlenbeck process. Show that an Ornstein-Uhlenbeck process has representation

$$X_t = e^{-\gamma t}\widetilde{B}\left(\frac{\sigma^2(e^{2\gamma t}-1)}{2\gamma}\right),$$

where $(\widetilde{B}_t)_{t\geq 0}$ is a Brownian motion started at $x$.
*Remark:* Note that the solution given in **Exercise 11-3** is a strong solution while the solution obtained here as a time-changed Brownian motion is a weak solution.

(c) Consider the SDEs

$$dX_t = \sigma(X_t)dB_t, \quad X_0 = x, \tag{3}$$

with $\sigma(x) > 0$ such that

$$G(t) = \int_0^t \frac{ds}{\sigma^2(B_s)}$$

is finite for finite $t$, and increases to infinity and $G(\infty) = \infty$ a.s.
Under this assumptions, $(G_t)_{t\geq 0}$ is adapted, continuous and strictly increasing to $G(\infty) = \infty$. Therefore its inverse is well defined:

$$\tau_t := G_t^{(-1)}.$$

Show that the process $X_t = B_{\tau_t}$ is a weak solution to the SDE (3).
*Hint:* Observe that for each $t$, $\tau_t$ is a stopping time and that $(\tau_t)_{t\geq 0}$ is increasing and show that $X_t = B_{\tau_t}$ is the solution of the martingale problem associated to (3).

**Exercise 14.2** Let $X$ be a Lévy process in $\mathbb{R}^d$ and $f_t(u) = E[e^{i\,u^{\mathrm{tr}} X_t}]$.

(a) Show that $X$ is stochastically continuous, i.e., for all $t$, $X_t$ is continuous in probability.

(b) Show that $f_{t+s}(u) = f_t(u)f_s(u)$ for all $s, t \geq 0$ and $f_0(u) = 1$ for any $u \in \mathbb{R}^d$.

(c) Use **b)** to show that $f_r(u) = f_1(u)^r$ for all *rational* $r \geq 0$.

(d) Show that $t \mapsto f_t(u)$ is right-continuous and conclude that $f_t(u) = f_1(u)^t$ and that $f_t(u) \neq 0$ for all $t \geq 0$ and $u \in \mathbb{R}^d$.

(e) Let $d = 1$. If $E[|X_1|] < \infty$, then $E[X_t] = tE[X_1]$ for all $t \geq 0$.

**Exercise 14.3**

(a) Let $N$ be a one-dimensional Poisson process and $(Y_i)_{i\geq 1}$ i.i.d. $\mathbb{R}^d$-valued random variables independent of $N$. We define the *compound Poisson process* by $X_t := \sum_{i=1}^{N_t} Y_i$. Show that $X$ is a Lévy process and calculate its Lévy triplet.

(b) Is there a Lévy process $X$ such that $X_1$ is uniformly distributed on $[0, 1]$?

(c) Let $X$ and $Y$ be both Lévy processes with respect to a filtration $(\mathcal{F}_t)$. Show that if $E[e^{i\,u^{\mathrm{tr}} X_t} e^{i\,v^{\mathrm{tr}} Y_t}] = E[e^{i\,u^{\mathrm{tr}} X_t}]\,E[e^{i\,v^{\mathrm{tr}} Y_t}]$ for all $u, v \in \mathbb{R}^d$ and $t \geq 0$, then $X$ and $Y$ are independent.