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Brownian Motion and Stochastic Calculus

Exercise sheet 14

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than June 9th

Exercise 14.1

(a) *Change of time in SDEs* Let $(f_t)_{t\geq 0}$ be an adapted, positive, increasing, differentiable process starting from zero and consider the following SDE

$$dX_t = \sqrt{f_t'}dB_t. \tag{1}$$

Show that the process B_{f_t} is a <u>weak</u> solution of (1).

Remark: In other words, given a Brownian motion $(B_t)_{t\geq 0}$ and a function f satisfying the previous assumptions, there exist a Brownian motion $(\widehat{B})_{t\geq 0}$, such that

$$d\widehat{B}_{f_t} = \sqrt{f_t'}dB_t.$$

(b) Recall from **Exercise 11-3** that a solution of the SDEs

$$dX_t = -\gamma X_t dt + \sigma dB_t, \quad X_0 = x, \tag{2}$$

is called Ornstein-Uhlenbeck process. Show that an Ornstein-Uhlenbeck process has representation

$$X_t = e^{-\gamma t}\widetilde{B}\left(\frac{\sigma^2(e^{2\gamma t}-1)}{2\gamma}\right),$$

where $(\widetilde{B}_t)_{t\geq 0}$ is a Brownian motion started at x.
Remark: Note that the solution given in **Exercise 11-3** is a strong solution while the solution obtained here as a time-changed Brownian motion is a weak solution.

(c) Consider the SDEs

$$dX_t = \sigma(X_t)dB_t, \quad X_0 = x, \tag{3}$$

with $\sigma(x) > 0$ such that

$$G(t) = \int_0^t \frac{ds}{\sigma^2(B_s)}$$

is finite for finite t, and increases to infinity and $G(\infty) = \infty$ a.s.
Under this assumptions, $(G_t)_{t\geq 0}$ is adapted, continuous and strictly increasing to $G(\infty) = \infty$. Therefore its inverse is well defined:

$$\tau_t := G_t^{(-1)}.$$

Show that the process $X_t = B_{\tau_t}$ is a weak solution to the SDE (3).
Hint: Observe that for each t, τ_t is a stopping time and that $(\tau_t)_{t\geq 0}$ is increasing and show that $X_t = B_{\tau_t}$ is the solution of the martingale problem associated to (3).

Solution 14.1

(a) We know that the process

$$X_t = \int_0^t \sqrt{f_t'} dB_t$$

is a local martingale with quadratic variation $\langle X \rangle_t = \int_0^t f_s' ds = f_t$. Denote by $\tau_t = f_t^{(-1)}$ the inverse of f. Accordingly to Theorem (7.66) the process $X(f_t^{(-1)}) = \widehat{B}_t$ is a Brownian motion wrt $\mathcal{F}_{\tau_t}$ and

$$X_t = \widehat{B}_{f_t}.$$

(b) With

$$f_t = \sigma^2 \frac{e^{2\gamma t} - 1}{2\gamma},$$

the process $\widetilde{B}(f_t)$ is a weak solution to the SDE

$$dY_t = \sigma e^{\gamma t} d\widetilde{B}_t.$$

Moreover $X_t = e^{-\gamma t} Y_t$. Indeed, integrating by parts,

$$dX_t = -\gamma X_t dt + \sigma d\widetilde{B}_t.$$

To have $X_0 = x$, take $(\widetilde{B}_t)_{t \geq 0}$ to be a Brownian motion started at x.

(c) The operator associated to (5) is given by

$$Lf(x) = \frac{1}{2}\sigma^2(x) f''(x).$$

We want to show that $X_t = B(\tau_t)$ is a solution to the martingale problem for L. Take $f \in C_0^2$, then we know that the process

$$M_t := f(B_t) - \int_0^t \frac{1}{2} f''(B_s) ds$$

is a martingale. Moreover $(\tau_t)_{t \geq 0}$ is an increasing sequence of stopping times and so (by OST) the process $M(\tau_t)$ is a martingale. Now we want the find an explicit expression for the process (τ_t). Using the formula for the derivative of the inverse function,

$$(G^{(-1)})_t' = \frac{1}{G'(G_t^{(-1)})} = \frac{1}{\sigma^2(B(G^{(-1)})_t)} = \sigma^2(B_{\tau_t}). \quad (4)$$

From (4) we see that $(\tau_t)_{t \geq 0}$ satisfies $d\tau_t = \sigma^2(B_{\tau_t}) dt$. Now perform a change of variable $s = \tau_u$ to obtain that the process

$$f(B_{\tau_t}) - \int_0^t \frac{1}{2}\sigma^2(B_{\tau_u}) f''(X_u) du$$

is a martingale and so $(X_t)_{t \geq 0}$ solves the martingale problem for L.

Alternative proof (some details are missing) From **a)** we know that there exists a Brownian motion $(\widehat{B}_t)_{t \geq 0}$ such that

$$dX_t = dB(G_t^{(-1)}) = \sqrt{(G^{(-1)})_t'} d\widehat{B}_t.$$

In particular, using the fact that $d\tau_t = \sigma^2(B_{\tau_t}) dt$, we get

$$dB(\tau_t) = \sigma^2(B(\tau_t)) d\widehat{B}_t.$$

Exercise 14.2 Let X be a Lévy process in $\mathbb{R}^d$ and $f_t(u) = E[e^{i\,u^{\mathrm{tr}} X_t}]$.

(a) Show that X is stochastically continuous, i.e., for all t, X_t is continuous in probability.

(b) Show that $f_{t+s}(u) = f_t(u) f_s(u)$ for all $s, t \geq 0$ and $f_0(u) = 1$ for any $u \in \mathbb{R}^d$.

(c) Use **b)** to show that $f_r(u) = f_1(u)^r$ for all *rational* $r \geq 0$.

(d) Show that $t \mapsto f_t(u)$ is right-continuous and conclude that $f_t(u) = f_1(u)^t$ and that $f_t(u) \neq 0$ for all $t \geq 0$ and $u \in \mathbb{R}^d$.

(e) Let $d = 1$. If $E[|X_1|] < \infty$, then $E[X_t] = tE[X_1]$ for all $t \geq 0$.

Solution 14.2

(a) Note that $P[|X_{t+h} - X_t| > c] = P[|X_{|h|}| > c] \to 0$ as $|h| \to 0$, because X is RC a.s.

(b) $f_0(u) = 1$ is clear. Using independence and stationarity of the increments as well as $X_0 = 0$ P-a.s., we have for any $s, t \geq 0$

$$\begin{aligned} f_{s+t}(u) &= E[\exp(iu^{\mathrm{tr}} X_{s+t})] = E[\exp(iu^{\mathrm{tr}}(X_{s+t} - X_s)) \exp(iu^{\mathrm{tr}} X_s)] \\ &= E[\exp(iu^{\mathrm{tr}}(X_{s+t} - X_s))] E[\exp(iu^{\mathrm{tr}} X_s)] = E[\exp(iu^{\mathrm{tr}} X_t)] E[\exp(iu^{\mathrm{tr}} X_s)] \\ &= f_t(u) f_s(u). \end{aligned}$$

(c) Let $m, n \in \mathbb{N}$. Using the property $f_{s+t}(u) = f_s(u) f_t(u)$ inductively, it follows that

$$(f_{m/n}(u))^n = f_m(u) = f_1(u)^m.$$

Hence, $f_{m/n}(u) = f_1(u)^{m/n}$.

(d) Right-continuity of $t \mapsto f_t(u)$ follows immediately from right-continuity of X and the bounded convergence theorem. Moreover, the function $t \mapsto f_1(u)^t$ is continuous and by part **c)**, $f_t(u) = f_1(u)^t$ for all $t \in \mathbb{Q}_+$. It follows that $f_t(u) = f_1(u)^t$ for all $t \geq 0$. Now, assume $f_t(u) = 0$ for some $t > 0$ and $u \in \mathbb{R}^d$. Then it follows that $f_{t/n}(u) = f_t(u)^{1/n} = 0$ for all $n \in \mathbb{N}$. Taking $n \to \infty$, we obtain a contradiction to the right-continuity.

(e) The integrability implies that the characteristic function is differentiable in u, and then $X_t \in L^1$ for all t. Moreover,

$$i\,E[X_t] = \partial_u f_t(u)|_{u=0} = t\psi'(u) e^{t\psi(u)}|_{u=0} = t\psi'(0) = i\,tE[X_1].$$

Exercise 14.3

(a) Let N be a one-dimensional Poisson process and $(Y_i)_{i\geq 1}$ i.i.d. $\mathbb{R}^d$-valued random variables independent of N. We define the *compound Poisson process* by $X_t := \sum_{i=1}^{N_t} Y_i$. Show that X is a Lévy process and calculate its Lévy triplet.

(b) Is there a Lévy process X such that X_1 is uniformly distributed on $[0,1]$?

(c) Let X and Y be both Lévy processes with respect to a filtration $(\mathcal{F}_t)$. Show that if $E[e^{i\,u^{\mathrm{tr}}X_t}e^{i\,v^{\mathrm{tr}}Y_t}] = E[e^{i\,u^{\mathrm{tr}}X_t}]\,E[e^{i\,v^{\mathrm{tr}}Y_t}]$ for all $u,v\in\mathbb{R}^d$ and $t\geq 0$, then X and Y are independent.

Solution 14.3

(a) We first show the independence of the increments. Let $n\in\mathbb{N}$ $0\leq t_0 < t_1 ... < t_n$ and $(f_i)_{i=1}^n$ Borel measurable functions. We need to show that

$$E\Big[\prod_{i=1}^n f_i(X_{t_i}-X_{t_{i-1}})\Big] = \prod_{i=1}^n E\big[f_i(X_{t_i}-X_{t_{i-1}})\big].$$

Let $\mathcal{G}$ the σ-field generated by $(N_t)_{t\geq 0}$. Then, by independence of the (Y_j) to N, we obtain that

$$\begin{aligned}E\Big[\prod_{i=1}^n f_i(X_{t_i}-X_{t_{i-1}})\Big] &= E\Big[\prod_{i=1}^n f_i\Big(\sum_{j=N_{t_{i-1}}+1}^{N_{t_i}} Y_j\Big)\Big]\\ &= E\Big[E\Big[\prod_{i=1}^n f_i\Big(\sum_{j=N_{t_{i-1}}+1}^{N_{t_i}} Y_j\Big)\Big|\,\mathcal{G}\Big]\Big]\\ &= E\Big[E\Big[\prod_{i=1}^n f_i\Big(\sum_{j=n_{i-1}+1}^{n_i} Y_j\Big)\Big]\Big|_{n_i=N_{t_i},n_{i-1}=N_{t_{i-1}}}\Big].\end{aligned}$$

Now, since the (Y_j) are i.i.d. we obtain that

$$\begin{aligned}E\Big[E\Big[\prod_{i=1}^n f_i\Big(\sum_{j=n_{i-1}+1}^{n_i} Y_j\Big)\Big]\Big|_{n_i=N_{t_i},n_{i-1}=N_{t_{i-1}}}\Big] &= E\Big[\prod_{i=1}^n E\Big[f_i\Big(\sum_{j=n_{i-1}+1}^{n_i} Y_j\Big)\Big]\Big|_{n_i=N_{t_i},n_{i-1}=N_{t_{i-1}}}\Big]\\ &= E\Big[\prod_{i=1}^n E\Big[f_i\Big(\sum_{j=1}^{n_i-n_{i-1}} Y_j\Big)\Big]\Big|_{n_i=N_{t_i},n_{i-1}=N_{t_{i-1}}}\Big]\\ &= E\Big[\prod_{i=1}^n E\Big[f_i\Big(\sum_{j=1}^{m_i} Y_j\Big)\Big]\Big|_{m_i=N_{t_i}-N_{t_{i-1}}}\Big].\end{aligned}$$

As (N_t) has independent increments we obtain that

$$\begin{aligned}E\Big[\prod_{i=1}^n E\Big[f_i\Big(\sum_{j=1}^{m_i} Y_j\Big)\Big]\Big|_{m_i=N_{t_i}-N_{t_{i-1}}}\Big] &= E\Big[\prod_{i=1}^n E\Big[f_i\Big(\sum_{j=1}^{m_i} Y_j\Big)\Big]\Big|_{m_i=N_{t_i}-N_{t_{i-1}}}\Big]\\ &= \prod_{i=1}^n E\Big[E\Big[f_i\Big(\sum_{j=n_{i-1}+1}^{n_i} Y_j\Big)\Big]\Big|_{n_i=N_{t_i},n_{i-1}=N_{t_{i-1}}}\Big]\\ &= \prod_{i=1}^n E\big[f_i(X_{t_i}-X_{t_{i-1}})\big].\end{aligned}$$

Now, we show that X has stationary increments. As X has independent increments, it's enough to show that for $s < t$ and f Borel, we have

$$E\big[f(X_t - X_s)\big] = E\big[f(X_{t-s})\big].$$

(see the solution of Exercise **6-3 a)** for details). With the same arguments we used for showing the independence of the increments of X, using that (N_t) has stationary increments, we obtain that

$$\begin{aligned}
E\big[f(X_t - X_s)\big] = E\Big[f\Big(\sum_{j=N_s+1}^{N_t} Y_j\Big)\Big] &= E\Big[E\Big[f\Big(\sum_{j=N_s+1}^{N_t} Y_j\Big)\Big|\mathcal{G}\Big]\Big] \\
&= E\Big[E\Big[f\Big(\sum_{j=n_s+1}^{n_t} Y_j\Big)\Big]\Big|_{n_t=N_t, n_s=N_s}\Big] \\
&= E\Big[E\Big[f\Big(\sum_{j=1}^{n_t-n_s} Y_j\Big)\Big]\Big|_{n_t=N_t, n_s=N_s}\Big] \\
&= E\Big[E\Big[f\Big(\sum_{j=1}^{m_{t,s}} Y_j\Big)\Big]\Big|_{m_{t,s}=N_t-N_s}\Big] \\
&= E\Big[E\Big[f\Big(\sum_{j=1}^{m_{t,s}} Y_j\Big)\Big]\Big|_{m_{t,s}=N_{t-s}}\Big] \\
&= E\Big[f\Big(\sum_{j=1}^{N_{t-s}} Y_j\Big)\Big] \\
&= E\big[f(X_{t-s})\big].
\end{aligned}$$

We conclude that X is a Lévy process. We proceed with calculating its triplet. For $u \in \mathbb{R}^d$,

$$\begin{aligned}
E[e^{i\,u^{\mathrm{tr}} X_t}] = E\Big[\sum_{k\geq 0} 1_{N_t=k} \prod_{j=1}^{k} e^{i\,u^{\mathrm{tr}} Y_j}\Big] &= \sum_{k\geq 0} P[N_t = k] E[e^{i\,u^{\mathrm{tr}} Y_1}]^k = \sum_{k\geq 0} e^{-\lambda t}\frac{(\lambda t)^k}{k!} E[e^{i\,u^{\mathrm{tr}} Y_1}]^k \\
&= e^{-\lambda t} \exp\Big(\lambda t E[e^{i\,u^{\mathrm{tr}} Y_1}]\Big) = \exp\Big(\lambda t \big\{E[e^{i\,u^{\mathrm{tr}} Y_1}] - 1\big\}\Big).
\end{aligned}$$

If F is the distribution of Y_1 and $\nu := \lambda F$, we have (for the truncation as in the lecture) the triplet $(b, 0, \nu)$, where $b = \int_{\{x:\,|x|\leq 1\}} x\,d\nu$.

(b) We show that any infinite divisible random variable X_1 with $\mathrm{supp}(X_1) \subseteq [a, b]$ for some $a < b$ is constant. This will imply that there is no Lévy process X with X_1 uniformly distributed on $[0, 1]$.
Let X_1 be an infinite divisible random variable with $\mathrm{supp}(X_1) \subseteq [a, b]$. By the infinite divisibility, we have $X_1 = \sum_{i=1}^{n} Y_i^n$ with $(Y_i^n)_{i=1}^n$ are i.i.d. This implies that $\mathrm{supp}(Y_i^n) \subseteq \big[\frac{a}{n}, \frac{b}{n}\big]$. Indeed, if e.g. $P\big[Y_i^n > \frac{b}{n}\big] > 0$, as $(Y_i^n)_{i=1}^n$ are i.i.d., we would have

$$P\big[X_1 > b\big] \geq P\Big[\bigcap_{i=1}^{n}\{Y_i^n > \frac{b}{n}\}\Big] = P\Big[Y_i^n > \frac{b}{n}\Big]^n > 0$$

which is a contradiction to the support of X_1 (In the same way as above one can show that $P\big[Y_i^n < \frac{a}{n}\big] = 0$).

(c) Let $n \in \mathbb{N}$ and $0 = t_0 < t_1 < ... < t_n$. We need to show that $(X_{t_1}, ..., X_{t_n})$ is independent of $(Y_{t_1}, ..., Y_{t_n})$. In the first step, we show that.

$$\begin{aligned}
&E\Big[\exp\Big(i\sum_{k=1}^{n} u_k^{\mathrm{tr}}(X_{t_k} - X_{t_{k-1}})\Big)\exp\Big(i\sum_{k=1}^{n} v_k^{\mathrm{tr}}(Y_{t_k} - Y_{t_{k-1}})\Big)\Big] \\
&= E\Big[\exp\Big(i\sum_{k=1}^{n} u_k^{\mathrm{tr}}(X_{t_k} - X_{t_{k-1}})\Big)\Big]E\Big[\exp\Big(i\sum_{k=1}^{n} v_k^{\mathrm{tr}}(Y_{t_k} - Y_{t_{k-1}})\Big)\Big].
\end{aligned}$$

We use an induction argument. For $n = 1$, this is the assumption. assume that it holds true for $n-1$. We obtain, as the increments are independent of the past, that

$$\begin{aligned}
&E\Big[\exp\Big(i\sum_{k=1}^{n} u_k^{\mathrm{tr}}(X_{t_k} - X_{t_{k-1}})\Big)\exp\Big(i\sum_{k=1}^{n} v_k^{\mathrm{tr}}(Y_{t_k} - Y_{t_{k-1}})\Big)\Big] \\
&= E\Big[\exp\Big(i\sum_{k=1}^{n-1} u_k^{\mathrm{tr}}(X_{t_k} - X_{t_{k-1}})\Big)\exp\Big(i\sum_{k=1}^{n-1} v_k^{\mathrm{tr}}(Y_{t_k} - Y_{t_{k-1}})\Big)\Big] \\
&\quad \cdot E\Big[\exp\Big(i\, u_n^{\mathrm{tr}}(X_{t_n} - X_{t_{n-1}})\Big)\exp\Big(i\, v_n^{\mathrm{tr}}(Y_{t_n} - Y_{t_{n-1}})\Big)\Big].
\end{aligned}$$

Now, using the induction hypothesis, we obtain that

$$\begin{aligned}
&= E\Big[\exp\Big(i\sum_{k=1}^{n-1} u_k^{\mathrm{tr}}(X_{t_k} - X_{t_{k-1}})\Big)\exp\Big(i\sum_{k=1}^{n-1} v_k^{\mathrm{tr}}(Y_{t_k} - Y_{t_{k-1}})\Big)\Big] \\
&\quad \cdot E\Big[\exp\Big(i\, u_n^{\mathrm{tr}}(X_{t_n} - X_{t_{n-1}})\Big)\exp\Big(i\, v_n^{\mathrm{tr}}(Y_{t_n} - Y_{t_{n-1}})\Big)\Big]. \\
&= E\Big[\exp\Big(i\sum_{k=1}^{n-1} u_k^{\mathrm{tr}}(X_{t_k} - X_{t_{k-1}})\Big)\Big]E\Big[\exp\Big(i\sum_{k=1}^{n-1} v_k^{\mathrm{tr}}(Y_{t_k} - Y_{t_{k-1}})\Big)\Big] \\
&\quad \cdot E\Big[\exp\Big(i\, u_n^{\mathrm{tr}}(X_{t_n} - X_{t_{n-1}})\Big)\exp\Big(i\, v_n^{\mathrm{tr}}(Y_{t_n} - Y_{t_{n-1}})\Big)\Big].
\end{aligned}$$

Using again the independence of increments of the past of $(\mathcal{F}_t)$, we obtain that

$$\begin{aligned}
&E\Big[\exp\Big(i\sum_{k=1}^{n-1} u_k^{\mathrm{tr}}(X_{t_k} - X_{t_{k-1}})\Big)\Big]E\Big[\exp\Big(i\sum_{k=1}^{n-1} v_k^{\mathrm{tr}}(Y_{t_k} - Y_{t_{k-1}})\Big)\Big] \\
&\quad \cdot E\Big[\exp\Big(i\, u_n^{\mathrm{tr}}(X_{t_n} - X_{t_{n-1}})\Big)\exp\Big(i\, v_n^{\mathrm{tr}}(Y_{t_n} - Y_{t_{n-1}})\Big)\Big] \\
&= E\Big[\exp\Big(i\sum_{k=1}^{n-1} u_k^{\mathrm{tr}}(X_{t_k} - X_{t_{k-1}})\Big)\Big]E\Big[\exp\Big(i\sum_{k=1}^{n-1} v_k^{\mathrm{tr}}(Y_{t_k} - Y_{t_{k-1}})\Big)\Big] \\
&\quad \cdot \frac{E\Big[\exp\Big(i\, u_n^{\mathrm{tr}} X_{t_n}\Big)\exp\Big(i\, v_n^{\mathrm{tr}} Y_{t_n}\Big)\Big]}{E\Big[\exp\Big(i\, u_n^{\mathrm{tr}} X_{t_{n-1}}\Big)\exp\Big(i\, v_n^{\mathrm{tr}} Y_{t_{n-1}}\Big)\Big]}.
\end{aligned}$$

Now, using the assumption that $E[e^{i\,u^{\mathrm{tr}}X_t}e^{i\,v^{\mathrm{tr}}Y_t}] = E[e^{i\,u^{\mathrm{tr}}X_t}]E[e^{i\,v^{\mathrm{tr}}Y_t}]$ for all $u, v \in \mathbb{R}^d$ and

$t \geq 0$, we obtain that

$$
\begin{aligned}
&E\Big[\exp\Big(i\sum_{k=1}^{n-1} u_k^{\mathrm{tr}}(X_{t_k}-X_{t_{k-1}})\Big)\Big]\,E\Big[\exp\Big(i\sum_{k=1}^{n-1} v_k^{\mathrm{tr}}(Y_{t_k}-Y_{t_{k-1}})\Big)\Big]\\
&\cdot\frac{E\Big[\exp\Big(i\,u_n^{\mathrm{tr}}X_{t_n}\Big)\exp\Big(i\,v_n^{\mathrm{tr}}Y_{t_n}\Big)\Big]}{E\Big[\exp\Big(i\,u_n^{\mathrm{tr}}X_{t_{n-1}}\Big)\exp\Big(i\,v_n^{\mathrm{tr}}Y_{t_{n-1}}\Big)\Big]}\\
=\;&E\Big[\exp\Big(i\sum_{k=1}^{n-1} u_k^{\mathrm{tr}}(X_{t_k}-X_{t_{k-1}})\Big)\Big]\,E\Big[\exp\Big(i\sum_{k=1}^{n-1} v_k^{\mathrm{tr}}(Y_{t_k}-Y_{t_{k-1}})\Big)\Big]\\
&\cdot\frac{E\Big[\exp\Big(i\,u_n^{\mathrm{tr}}X_{t_n}\Big)\Big]\,E\Big[\exp\Big(i\,v_n^{\mathrm{tr}}Y_{t_n}\Big)\Big]}{E\Big[\exp\Big(i\,u_n^{\mathrm{tr}}X_{t_{n-1}}\Big)\Big]\,E\Big[\exp\Big(i\,v_n^{\mathrm{tr}}Y_{t_{n-1}}\Big)\Big]}
\end{aligned}
$$

using twice the independence of increments of the past of $(\mathcal{F}_t)$ yields that

$$
\begin{aligned}
&E\Big[\exp\Big(i\sum_{k=1}^{n-1} u_k^{\mathrm{tr}}(X_{t_k}-X_{t_{k-1}})\Big)\Big]\,E\Big[\exp\Big(i\sum_{k=1}^{n-1} v_k^{\mathrm{tr}}(Y_{t_k}-Y_{t_{k-1}})\Big)\Big]\\
&\cdot\frac{E\Big[\exp\Big(i\,u_n^{\mathrm{tr}}X_{t_n}\Big)\Big]\,E\Big[\exp\Big(i\,v_n^{\mathrm{tr}}Y_{t_n}\Big)\Big]}{E\Big[\exp\Big(i\,u_n^{\mathrm{tr}}X_{t_{n-1}}\Big)\Big]\,E\Big[\exp\Big(i\,v_n^{\mathrm{tr}}Y_{t_{n-1}}\Big)\Big]}\\
=\;&E\Big[\exp\Big(i\sum_{k=1}^{n-1} u_k^{\mathrm{tr}}(X_{t_k}-X_{t_{k-1}})\Big)\Big]\,E\Big[\exp\Big(i\sum_{k=1}^{n-1} v_k^{\mathrm{tr}}(Y_{t_k}-Y_{t_{k-1}})\Big)\Big]\\
&\cdot E\Big[\exp\Big(i\,u_n^{\mathrm{tr}}\Big(X_{t_n}-X_{t_{n-1}})\Big)\Big]\,E\Big[\exp\Big(i\,v_n^{\mathrm{tr}}\Big(Y_{t_n}-Y_{t_{n-1}})\Big)\Big]\\
=\;&E\Big[\exp\Big(i\sum_{k=1}^{n} u_k^{\mathrm{tr}}(X_{t_k}-X_{t_{k-1}})\Big)\Big]E\Big[\exp\Big(i\sum_{k=1}^{n} v_k^{\mathrm{tr}}(Y_{t_k}-Y_{t_{k-1}})\Big)\Big].
\end{aligned}
$$

So we have proved the claim. Next, since characteristic functions determine the law of random vectors, we conclude that $\bar{X} := (X_{t_1}-X_{t_0},...,X_{t_n}-X_{t_{n-1}})$ and $\bar{Y} := (Y_{t_1}-X_{t_0},...,Y_{t_n}-Y_{t_{n-1}})$ are independent. Thus, fom the continuity theorem, we obtain that $f(\bar{X})$ and $f(\bar{Y})$ are independent for every f continuous. As $X_{t_0} = Y_{t_0} = 0$ we can find a linear (and hence continuous) function f such that $f(\bar{X}) = (X_{t_1}, X_{t_2},...,X_{t_n})$ and $f(\bar{Y}) = (Y_{t_1}, Y_{t_2},...,Y_{t_n})$. Thus we conclude that $(X_{t_1},...,X_{t_n})$ and $(Y_{t_1},...,Y_{t_n})$ are independent, which was to show.