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# Brownian Motion and Stochastic Calculus

## Exercise sheet 1

*Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than March 3rd*

**Exercise 1.1** Let $W$ be a Brownian motion on $[0,1]$ and define the *Brownian bridge* $X = (X_t)_{0 \le t \le 1}$ by $X_t = W_t - tW_1$.

(a) Show that $X$ is a Gaussian process and calculate its mean and covariance functions. Sketch a typical path of $X$.

(b) Show that $X$ does **not** have independent increments.

**Solution 1.1**

(a) We need to show that for any $n \ge 1$ and any $0 \le t_1 \le t_2 ... \le t_n \le 1$ the random vector $(X_{t_1}, ..., X_{t_n})$ is a Gaussian vector. However any linear combination of $X_{t_1}, ..., X_{t_n}$ is also a linear combination of $W_{t_1}, ..., W_{t_n}, W_1$ which is Gaussian because the Brownian motion is a Gaussian process.

For any $t \in [0,1]$ we have
$$E[X_t] = E[W_t - t\,W_1] = 0.$$
For any $0 \le s, t \le 1$, using that $\mathrm{Cov}(W_t, W_s) = t \wedge s$ (see Prop 1.15), we have
$$\begin{aligned}\mathrm{Cov}(X_t, X_s) &= \mathrm{Cov}(W_t - t\,W_1, W_s - s\,W_1)\\ &= \mathrm{Cov}(W_t, W_s) - s\,\mathrm{Cov}(W_t, W_1) - t\,\mathrm{Cov}(W_1, W_s) + ts\,\mathrm{Cov}(W_1, W_1)\\ &= t \wedge s - ts.\end{aligned}$$

(b) Take any $t \in (0,1)$. We show that the increment $X_1 - X_t$, $X_t - X_0$ are correlated. In the same way as above we obtain that
$$\mathrm{Cov}(X_1 - X_t,\, X_t - X_0) = \mathrm{Cov}(-W_t + t\,W_1,\, W_t - t\,W_1) = t(t-1) \neq 0.$$

**Exercise 1.2** Let $(\Omega, \mathcal{F}, P)$ be a probability space and assume that $X = (X_t)_{t \geq 0}$, $Y = (Y_t)_{t \geq 0}$ are two stochastic processes on $(\Omega, \mathcal{F}, P)$. Two processes $Z$ and $Z'$ on $(\Omega, \mathcal{F}, P)$ are said to be *modifications* of each other if $P(Z_t = Z'_t) = 1 \ \forall\ t \geq 0$, while $Z$ and $Z'$ are *indistinguishable* if $P(Z_t = Z'_t \ \forall\ t \geq 0) = 1$.

(a) Assume that $X$ and $Y$ are both right-continuous or both left-continuous. Show that the processes are modifications of each other if and only if they are indistinguishable.

**Remark:** A stochastic process is said to *have the path property* $\mathcal{P}$ ($\mathcal{P}$ can be continuity, right-continuity, differentiability, ...) if the property $\mathcal{P}$ holds for $P$-almost every path.

(b) Give an example showing that one of the implications of part **a)** does not hold for general $X$, $Y$.

**Solution 1.2**

(a) We just show that the fact that $X$ is a modification of $Y$ implies the indistinguishability, since the converse is obvious. Without loss of generality, we assume that $X$ and $Y$ are right-continuous.

For $t \geq 0$, we define the null set $N_t := \{\omega : X_t(\omega) \neq Y_t(\omega)\}$. We consider $N := \cup_{t \in \mathbb{Q}_+} N_t$, which remains a null set as a countable union of null sets. Finally, we introduce the null set $A_Z := \{\omega : Z_\cdot(\omega) \text{ not right-continuous}\}$ for $Z = X, Y$ and we define $M := A_X \cup A_Y \cup N$, which is still a null set.

It suffices to check that, for all $\omega \in M^c$, $X_t(\omega) = Y_t(\omega) \ \forall\ t \geq 0$. By definition of $M$ we clearly have that, for $\omega \in M^c$, $X_t(\omega) = Y_t(\omega) \ \forall\ t \in \mathbb{Q}_+$. Now, take any $t \geq 0$ and let $(t_n)$ be a sequence in $\mathbb{Q}_+$ with $t_n \downarrow t$. The right-continuity of the paths $X_\cdot(\omega)$ and $Y_\cdot(\omega)$ then implies $X_t(\omega) = \lim_{n\to\infty} X_{t_n}(\omega) = \lim_{n\to\infty} Y_{t_n}(\omega) = Y_t(\omega)$.

(b) Take $\Omega = [0, \infty)$, $\mathcal{F} = \mathcal{B}([0, \infty))$ the Borel $\sigma$-algebra, and $P$ a probability measure with $P(\{\omega\}) = 0 \ \forall\ \omega \in \Omega$ (for instance, the exponential distribution).

Set $X \equiv 0$ and $Y_t(\omega) = \begin{cases} 1, & t = \omega, \\ 0, & \text{else.} \end{cases}$

Then, $P[X_t = Y_t] = 1 \ \forall\ t \geq 0$, since single points have no mass, but $\{X_t = Y_t \ \forall\ t \geq 0\} = \emptyset$. Note that all sample paths of $X$ are continuous, while all sample paths of $Y$ are discontinuous at $t = \omega$.

**Exercise 1.3** Let $X = (X_t)_{t\geq 0}$ be a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. The aim of this exercise is to show the following chain of implications:

$$X \text{ optional} \Rightarrow X \text{ progressively measurable} \Rightarrow X \text{ product-measurable and adapted.}$$

(a) Show that every progressively measurable process is product-measurable and adapted.

(b) Assume that $X$ is adapted and *every* path of $X$ is right-continuous. Show that $X$ is progressively measurable.
*Remark:* The same conclusion holds true if every path of $X$ is left-continuous.
*Hint:* For fixed $t \geq 0$, consider an approximating sequence of processes $Y^n$ on $\Omega \times [0, t]$ given by $Y_0^n = X_0$ and $Y_u^n = \sum_{k=0}^{2^n-1} 1_{(tk2^{-n}, t(k+1)2^{-n}]}(u) X_{t(k+1)2^{-n}}$ for $u \in (0, t]$.

(c) Recall that the optional $\sigma$-field $\mathcal{O}$ is generated by the class $\overline{\mathcal{M}}$ of all adapted processes whose paths are all RCLL. Show that $\mathcal{O}$ is also generated by the subclass $\mathcal{M}$ of all *bounded* processes in $\overline{\mathcal{M}}$.

(d) Use the monotone class theorem to show that every optional process is progressively measurable.

**Solution 1.3**

(a) Let $X$ be progressively measurable. Then $X|_{\Omega\times[0,t]}$ is $\mathcal{F}_t \otimes \mathcal{B}[0, t]$-measurable for every $t \geq 0$. For any $t \geq 0$, we see that $X_t = X \circ i_t$, where $i_t : (\Omega, \mathcal{F}_t) \to (\Omega \times [0, t], \mathcal{F}_t \otimes \mathcal{B}[0, t])$, $\omega \mapsto (\omega, t)$ is measurable. Therefore, $X_t$ is $\mathcal{F}_t$-measurable for every $t \geq 0$. Moreover, the processes $X^n$ defined by $X_u^n := X|_{\Omega\times[0,n]} 1_{[0,n]}(u)$, $u \geq 0$, are $\mathcal{F} \otimes \mathcal{B}[0, \infty)$-measurable. Since $X^n \to X$ pointwise (in $(t, \omega)$) as $n \to \infty$, also $X$ is $\mathcal{F} \otimes \mathcal{B}[0, \infty)$-measurable.

(b) Fix a $t \geq 0$ and consider the sequence of processes $Y^n$ on $\Omega \times [0, t]$ given by $Y_0^n = X_0$ and $Y_u^n = \sum_{k=1}^{2^n-1} 1_{(tk2^{-n}, t(k+1)2^{-n}]}(u) X_{t(k+1)2^{-n}}$ for $u \in (0, t]$. By construction, each $Y^n$ is $\mathcal{F}_t \otimes \mathcal{B}[0, t]$-measurable. Since $Y^n \to X|_{\Omega\times[0,t]}$ pointwise as $n \to \infty$ due to right-continuity, the result follows.

(c) Let $X$ be adapted, with all paths being RCLL. Consider the processes $X^n := (X \wedge n) \vee (-n)$. Clearly, each $X^n$ is bounded and RCLL. Thus, each $X^n$ is $\sigma(\mathcal{M})$-measurable. As the pointwise limit of the $X^n$, also $X$ is $\sigma(\mathcal{M})$-measurable. It follows that $\mathcal{O} \subset \sigma(\mathcal{M})$. The reverse inclusion is trivial.

(d) If a process $X$ is optional, then $X^n := X\, 1_{\{|X|\leq n\}}$ is also optional and of course $X^n \to X$; so if each $X^n$ is progressively measurable, then so is $X$, and hence we can assume without loss of generality that $X$ is bounded.
Let $\mathcal{H}$ denote the real vector space of bounded, progressively measurable processes. By part b), $\mathcal{H}$ contains $\mathcal{M}$. Clearly, $\mathcal{H}$ contains the constant process 1 and is closed under monotone bounded convergence. Also, $\mathcal{M}$ is closed under multiplication. The monotone class theorem yields that every bounded $\sigma(\mathcal{M})$-measurable process is contained in $\mathcal{H}$. Due to c), we conclude that every bounded optional process is progressively measurable.