Coordinator
A. Sepúlveda

Brownian Motion and Stochastic Calculus

Exercise sheet 2

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than March 10th

Exercise 2.1 Let X be a real valued random variable with standard normal distribution as law and Y a random variable independent of X with law defined by

$$P[Y=1]=p \quad \text{and} \quad P[Y=-1]=1-p, \qquad (0 \leq p \leq 1).$$

We define $Z := XY$.

(a) What is the law of Z? Is the vector (X,Z) a Gaussian vector?

(b) Calculate $\mathrm{Cov}(X,Z)$. For which $p \in [0,1]$ are the random variables X and Z uncorrelated, i.e. $\mathrm{Cov}(X,Z)=0$?

(c) Show that for each $p \in [0,1]$, the random variables X and Z are **not** independent.

Exercise 2.2 Let $(\Omega, \mathcal{F}, P)$ be a probability space, W a Brownian motion on $[0,\infty)$, Z a random variable independent of W such that $P[Z=1]=P[Z=-1]=\frac{1}{2}$, and $t^* \in [0,\infty)$. We define another stochastic process $W' = (W'_t)_{t\geq 0}$ by

$$W'_t = W_t 1_{\{t<t^*\}} + \Big(W_{t^*} + Z(W_t - W_{t^*})\Big)1_{\{t\geq t^*\}}.$$

Show that W' is a Brownian motion.

Interpretation: W' is obtained from W by flipping an independent fair coin at t^* and reflecting the W-trajectories after t^* at the level W_{t^*} if head comes up.

Exercise 2.3 Let X be a stochastic process on a probability space $(\Omega, \mathcal{F}, P)$ with $X_0 = 0$ P-a.s., and let $\mathbb{F}^X = (\mathcal{F}^X_t)_{t\geq 0}$ denote the (raw) filtration generated by X, i.e., $\mathcal{F}^X_t = \sigma\left(X_s;\, s \leq t\right)$. Show that the following two properties are equivalent:

(i) X has *independent increments*, i.e., for all $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \cdots < t_n < \infty$, the increments $X_{t_i} - X_{t_{i-1}}$, $i = 1, \ldots, n$, are independent.

(ii) X has $\mathbb{F}^X$*-independent increments*, i.e., $X_t - X_s$ is independent of $\mathcal{F}^X_s$ whenever $t > s$.

Remark: This also shows the equivalence between the two definitions of Brownian motion with properties (BM2) and (BM2′), respectively.
Hint: For proving "(i) $\Rightarrow$ (ii)", you can use the monotone class theorem. When choosing $\mathcal{H}$, recall that a random variable Y is independent of a σ-algebra $\mathcal{G}$ if and only if one has the product formula $E[g(Y)Z] = E[g(Y)]E[Z]$ for all bounded Borel-measurable functions $g : \mathbb{R} \to \mathbb{R}$ and all bounded $\mathcal{G}$-measurable random variables Z.

Exercise 2.4 The objective of this problem is to prove that there exists some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and measurable function W from $(\Omega, \mathcal{F})$ to $(C[0,1], \mathcal{B}(C[0,1]))$ (The continuous functions with its Borel σ-algebra) such that W, under $\mathbb{P}$, has the law of a Brownian motion.

(a) Suppose that $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is big enough so it contains a sequence $(Y_{n,k})_{n,l\geq 0}$ i.i.d. standard normal. Show that $\omega \mapsto W_{\cdot}^{N}(\omega) := Y_{0,0}(\omega)\varphi_0(\cdot) + \sum_{n=0}^{N}\sum_{k=1}^{2^n} Y_{n,k}(\omega)\varphi_{n,k}(\cdot)$ is a measurable function from $(\tilde{\Omega}, \tilde{\mathcal{F}})$ to $(C[0,1], \mathcal{B}(C[0,1]))$. Here $\varphi_{n,k}$ are the Schauder functions.

(b) Show that there exist a measurable subset of $\Omega \subseteq \tilde{\Omega}$ with $\tilde{\mathbb{P}}(\Omega) = 1$ such that for all $\omega \in \Omega$, $W_{\cdot}(\omega)^N \to W_{\cdot}(\omega)$ as $N \to \infty$ in the topology of $C[0,1]$. Conclude that $\omega \mapsto W_{\cdot}(\omega)$ is a measurable function from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(C[0,1], \mathcal{B}(C[0,1]))$, where $\mathcal{F}$ and $\mathbb{P}$ are the restriction of $\tilde{\mathcal{F}}$ and $\tilde{\mathbb{P}}$ to Ω, and that the law of W_t is that of a Brownian Motion.