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Brownian Motion and Stochastic Calculus

Exercise sheet 2

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than March 10th

Exercise 2.1 Let X be a real valued random variable with standard normal distribution as law and Y a random variable independent of X with law defined by

$$P[Y=1]=p \quad \text{and} \quad P[Y=-1]=1-p, \qquad (0 \le p \le 1).$$

We define $Z := XY$.

(a) What is the law of Z? Is the vector (X,Z) a Gaussian vector?

(b) Calculate $\mathrm{Cov}(X,Z)$. For which $p \in [0,1]$ are the random variables X and Z uncorrelated, i.e. $\mathrm{Cov}(X,Z)=0$?

(c) Show that for each $p \in [0,1]$, the random variables X and Z are **not** independent.

Solution 2.1

(a) We show that $Z \sim \mathcal{N}(0,1)$ by calculating its characteristic function. Using the independence of X and Y and that X and $-X \sim \mathcal{N}(0,1)$, we get for each $t \in \mathbb{R}$ that

$$\begin{aligned}\varphi_Z(t) := E\big[e^{itZ}\big] &= E\big[e^{itX}\,1_{\{Y=1\}}\big] + E\big[e^{-itX}\,1_{\{Y=-1\}}\big]\\ &= E\big[e^{itX}\big]\;P\big[Y=1\big] + E\big[e^{-itX}\big]\;P\big[Y=-1\big]\\ &= e^{-\frac{1}{2}t^2}.\end{aligned}$$

To prove that (X,Z) is a Gaussian vector, we need to show that for any $\lambda_1, \lambda_2 \in \mathbb{R}$, the random variable $\lambda_1 X + \lambda_2 Z$ is normal distributed. Fix any $\lambda_1, \lambda_2 \in \mathbb{R}$.
For $p \in \{0,1\}$ we see that

$$\lambda_1 X + \lambda_2 Z = cX$$

for some $c \in \mathbb{R}$. Therefore, as $X \sim \mathcal{N}(0,1)$ we get that $\lambda_1 X + \lambda_2 Z \sim \mathcal{N}(0,c^2)$ and thus (X,Z) is a Gaussian vector.
Now, let $p \in [0,1] \setminus \{0,1\}$. Assume by contradiction that (X,Z) is a Gaussian vector. Then $X+Z$ is normal distributed. But since $P[X=0]=0$ as X is normal distributed, we get that

$$P\big[X+Z=0\big] = P\big[Y=-1\big] = 1-p \neq 0$$

which gives a contradiction. We conclude that

$$(X,Z) \text{ is a Gaussian vector } \iff p \in \{0,1\}.$$

(b) Using that $X \sim \mathcal{N}(0,1)$, the independence of X and Y and that $E[Y]=2p-1$, we get

$$\begin{aligned}\mathrm{Cov}\big(X,Z\big) &= E\big[X^2\,Y\big] - E\big[X\big]\;E\big[X\,Y\big]\\ &= E\big[X^2\big]\;E\big[Y\big]\\ &= \mathrm{Var}\big(X\big)\;E\big[Y\big]\\ &= 2p-1.\end{aligned}$$

Therefore,

$$\mathrm{Cov}\big(X,Z\big) = 0 \iff p = 1/2.$$

(c) Assume by contradiction that X and Z are independent. Then, as $Z \sim \mathcal{N}(0,1)$,

$$0 = P\Big[|Z| > 1 \,\Big|\, |X| \leq 1\Big] = P\Big[|Z| > 1\Big] \neq 0$$

which gives a contradiction.
Alternative proof: For $p \in (0,1)$, if X and Z were independent, since by a) X and Z are normal distributed, (X,Z) would be a Gaussian vector, which is a contradiction to a). For $p \in \{0,1\}$, it is clear that we do not have independence, since in that case

$$X = Z \;\text{ a.s.} \quad \text{or} \quad X = -Z \;\text{ a.s.}$$

Exercise 2.2 Let $(\Omega, \mathcal{F}, P)$ be a probability space, W a Brownian motion on $[0, \infty)$, Z a random variable independent of W such that $P[Z = 1] = P[Z = -1] = \frac{1}{2}$, and $t^* \in [0, \infty)$. We define another stochastic process $W' = (W'_t)_{t \geq 0}$ by

$$W'_t = W_t 1_{\{t<t^*\}} + \Big(W_{t^*} + Z(W_t - W_{t^*})\Big) 1_{\{t \geq t^*\}}.$$

Show that W' is a Brownian motion.

Interpretation: W' is obtained from W by flipping an independent fair coin at t^* and reflecting the W-trajectories after t^* at the level W_{t^*} if head comes up.

Solution 2.2 It is clear that $\mathbb{P}(W'_0 = 0) = 1$ and that W' is $\mathbb{P}$-a.s. continuous. It is only left to prove that it has normal independent increments with the correct variance. To do that take $0 \leq t_0 < .. < t_k \leq t^* < t_{k+1} < .. < t_n$ and see that the characteristic function of the random vector $V := (W'_{t_0} - W'_{t_1}, ..., W'_{t_n} - W'_{t_{n-1}})$, $\varphi_V(\lambda_1, ..., \lambda_n)$, is

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(i\sum_{j=1}^n \lambda_j (W'_{t_j} - W'_{t_{j-1}})\right)\right] \\
&= \mathbb{E}\left[e^{i\lambda_{k+1}(W'_{t^*} - W'_{t_k}) + \sum_{j=1}^k i\lambda_j (W'_{t_j} - W'_{t_{j-1}})}\right] \mathbb{E}\left[e^{iZ\left(\lambda_{k+1}(W'_{t_{k+1}} - W'_{t^*}) + \sum_{j=k+2}^n \lambda_i (W'_{t_j} - W'_{t_{j-1}})\right)}\right] \\
&= \exp\left(-\frac{1}{2}\lambda_{k+1}^2 (t^* - t_k) - \frac{1}{2}\sum_{j=1}^k \lambda_j^2 (t_j - t_{j-1})\right) \mathbb{E}\left[e^{iZ\left(\lambda_{k+1}(W'_{t_{k+1}} - W'_{t^*}) + \sum_{j=k+2}^n \lambda_j (W'_{t_j} - W'_{t_{j-1}})\right)}\right],
\end{aligned}$$

where we have used that all coordinates of V are independent and that V is independent from Z. And that the characteristic function of a centred normal random variable with variance σ^2 is $\varphi(\lambda) = \exp(-\lambda^2\sigma^2/2)$. To conclude note that

$$\begin{aligned}
&\mathbb{E}\left[e^{iZ\left(\lambda_{k+1}(W'_{t_{k+1}} - W'_{t^*}) + \sum_{j=k+2}^n \lambda_j (W'_{t_j} - W'_{t_{j-1}})\right)}\right] \\
&= \frac{1}{2}\mathbb{E}\left[e^{i\lambda_{k+1}(W'_{t_{k+1}} - W'_{t^*}) + i\sum_{j=k+2}^n \lambda_j (W'_{t_j} - W'_{t_{j-1}})}\right] + \frac{1}{2}\mathbb{E}\left[e^{-i\lambda_{k+1}(W'_{t_{k+1}} - W'_{t^*}) - \sum_{j=k+2}^n i\lambda_j (W'_{t_j} - W'_{t_{j-1}})}\right] \\
&= \exp\left(-\frac{1}{2}\lambda_{k+1}^2 (t_{k+1} - t^*) - \frac{1}{2}\sum_{j=k+2}^n \lambda_j^2 (t_j - t_{j-1})\right).
\end{aligned}$$

Thus,

$$\varphi_V(\lambda_1, ..., \lambda_n) = \exp\left(-\frac{1}{2}\sum_{j=1}^n \lambda_j^2 (t_j - t_{j-1})\right)$$

which is exactly the characteristic function of centred independent normal variables with the required variance.

Exercise 2.3 Let X be a stochastic process on a probability space $(\Omega, \mathcal{F}, P)$ with $X_0 = 0$ P-a.s., and let $\mathbb{F}^X = (\mathcal{F}_t^X)_{t\geq 0}$ denote the (raw) filtration generated by X, i.e., $\mathcal{F}_t^X = \sigma(X_s;\, s \leq t)$. Show that the following two properties are equivalent:

(i) X has *independent increments*, i.e., for all $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \cdots < t_n < \infty$, the increments $X_{t_i} - X_{t_{i-1}}$, $i = 1, \ldots, n$, are independent.

(ii) X has $\mathbb{F}^X$*-independent increments*, i.e., $X_t - X_s$ is independent of $\mathcal{F}_s^X$ whenever $t > s$.

Remark: This also shows the equivalence between the two definitions of Brownian motion with properties (BM2) and (BM2′), respectively.
Hint: For proving "(i) ⇒ (ii)", you can use the monotone class theorem. When choosing $\mathcal{H}$, recall that a random variable Y is independent of a σ-algebra $\mathcal{G}$ if and only if one has the product formula $E[g(Y)Z] = E[g(Y)]E[Z]$ for all bounded Borel-measurable functions $g : \mathbb{R} \to \mathbb{R}$ and all bounded $\mathcal{G}$-measurable random variables Z.

Solution 2.3 First, assume that X has independent increments and fix $t \geq s \geq 0$. The family

$$\mathcal{M} = \left\{ \prod_{i=1}^{n} h_i(X_{s_i}) : n \in \mathbb{N}, 0 \leq s_1 < \cdots < s_n \leq s, h_i : \mathbb{R} \to \mathbb{R} \text{ Borel and bounded} \right\}$$

of bounded, real-valued functions on Ω is closed under multiplication. Moreover, note that $\sigma(\mathcal{M}) = \mathcal{F}_s^X$. Let $\mathcal{H}$ denote the real vector space of all bounded, real-valued, $\mathcal{F}_s^X$-measurable functions Z on Ω with the property that:

$$E[g(X_t - X_s)Z] = E[g(X_t - X_s)]E[Z] \quad \text{for all bounded Borel functions } g : \mathbb{R} \to \mathbb{R}.$$

Clearly, $\mathcal{H}$ contains the constant function 1 and is closed under monotone bounded convergence (we even do not need monotonicity).

Next, we show that $\mathcal{H}$ contains $\mathcal{M}$. Fix a typical element $Z = \prod_{i=1}^n h_i(X_{s_i})$ in $\mathcal{M}$. Define the function $h : \mathbb{R}^n \to \mathbb{R}$ by $h(x) = \prod_{i=1}^n h_i(x_i)$ where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Note that h is again a Borel function. Then we can write $Z = h(X_{s_1}, \ldots, X_{s_n})$. Since $X_0 = 0$ P-a.s., we also have

$$(X_{s_1}, \ldots, X_{s_n}) = f(X_{s_1} - X_0, X_{s_2} - X_{s_1}, \ldots, X_{s_n} - X_{s_{n-1}}) \quad P\text{-a.s.}$$

for a linear transformation f. Finally,

$$\begin{aligned} E[g(X_t - X_s)Z] &= E[g(X_t - X_s)h(X_{s_1}, \ldots, X_{s_n})] \\ &= E[g(X_t - X_s)(h \circ f)(X_{s_1} - X_0, X_{s_2} - X_{s_1}, \ldots, X_{s_n} - X_{s_{n-1}})] \\ &= E[g(X_t - X_s)]E[(h \circ f)(X_{s_1} - X_0, X_{s_2} - X_{s_1}, \ldots, X_{s_n} - X_{s_{n-1}})] \\ &= E[g(X_t - X_s)]E[Z] \end{aligned}$$

where we use the assumption that $X_t - X_s$ is independent of $(X_{s_1} - X_0, X_{s_2} - X_{s_1}, \ldots, X_{s_n} - X_{s_{n-1}})$ in the third equality. Thus, $Z \in \mathcal{H}$.

The monotone class theorem yields that $\mathcal{H}$ contains every bounded $\mathcal{F}_s^X$-measurable function on Ω. In particular, $X_t - X_s$ is independent of $\mathcal{F}_s^X$.

For the converse implication, we proceed by induction on n. The case $n = 1$ is trivial, so fix $n \geq 2$, $0 \leq t_0 < t_1 < \ldots < t_n < \infty$, and $A_i \in \mathcal{B}(\mathbb{R})$, $i = 1, \ldots, n$. Conditioning on $\mathcal{F}_{t_{n-1}}^X$, and using (ii) for $t = t_n$ and $s = t_{n-1}$, we obtain

$$P\left[\bigcap_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})^{-1}(A_i)\right] = P\left[\bigcap_{i=1}^{n-1} (X_{t_i} - X_{t_{i-1}})^{-1}(A_i)\right] P\left[(X_{t_n} - X_{t_{n-1}})^{-1}(A_n)\right].$$

Applying the induction hypothesis to the first factor on the right-hand side completes the proof.

Exercise 2.4 The objective of this problem is to prove that there exists some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and measurable function W from $(\Omega, \mathcal{F})$ to $(C[0,1], \mathcal{B}(C[0,1])$ (The continuous functions with its Borel σ-algebra) such that W, under $\mathbb{P}$, has the law of a Brownian motion.

(a) Suppose that $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is big enough so it contains a sequence $(Y_{n,k})_{n,l \geq 0}$ i.i.d. standard normal. Show that $\omega \mapsto W_\cdot^N(\omega) := Y_{0,0}(\omega)\varphi_0(\cdot) + \sum_{n=0}^{N} \sum_{k=1}^{2^n} Y_{n,k}(\omega)\varphi_{n,k}(\cdot)$ is a measurable function from $(\tilde{\Omega}, \tilde{\mathcal{F}})$ to $(C[0,1], \mathcal{B}(C[0,1]))$. Here $\varphi_{n,k}$ are the Schauder functions.

(b) Show that there exist a measurable subset of $\Omega \subseteq \tilde{\Omega}$ with $\tilde{\mathbb{P}}(\Omega) = 1$ such that for all $\omega \in \Omega$, $W_\cdot(\omega)^N \to W_\cdot(\omega)$ as $N \to \infty$ in the topology of $C[0,1]$. Conclude that $\omega \mapsto W_\cdot(\omega)$ is a measurable function from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(C[0,1], \mathcal{B}(C[0,1]))$, where $\mathcal{F}$ and $\mathbb{P}$ are the restriction of $\tilde{\mathcal{F}}$ and $\tilde{\mathbb{P}}$ to Ω, and that the law of W_t is that of a Brownian Motion.

Solution 2.4

(a) Note that the functions $\omega \mapsto \varphi_{n,k}$ and $\omega \mapsto Y_{n,k}(\omega)$ are measurable. So W_t^N is also measurable.

(b) In Theorem (5.11) of the script it has been shown that $\tilde{\mathbb{P}}$-a.s. $W_\cdot^N(\omega)$ converges uniformly to some $W_t^\infty(\omega)$. By definition of $\mathbb{P}$-a.s., we have that measurable subset of $\Omega \subseteq \tilde{\Omega}$ with $\tilde{\mathbb{P}}(\omega) = 1$ such that for all $\omega \in \Omega$, $W_\cdot^N(\omega) \to W_\cdot^\infty(\omega)$ as $N \to \infty$ in the topology of $C[0,1]$. Under $(\Omega, \mathcal{F}, \mathbb{P})$ we also have that $W_\cdot^N$ are measurable functions, thus W_t^∞ is a measurable function with the law of a BM thanks to Theorem 5.11 and the fact that for any measurable event A, $\tilde{\mathbb{P}}(A) = \mathbb{P}(A \cap \tilde{\Omega})$.