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Brownian Motion and Stochastic Calculus

Exercise sheet 3

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than March 17th

Exercise 3.1 Let $X = (X^1, \dots, X^d)$ be an $\mathbb{R}^d$-valued stochastic process on $[0,1]$. For $x = (x^1, \dots, x^d) \in \mathbb{R}^d$ with Euclidean norm $\|x\| = 1$, we define the process $Y^x = (Y^x_t)_{0 \le t \le 1}$ by

$$Y^x_t = x^\top X_t = \sum_{i=1}^d x^i X^i_t.$$

Prove that if X is a Brownian motion in $\mathbb{R}^d$, then every Y^x is a Brownian motion in $\mathbb{R}$.

Exercise 3.2 Let $(B_t)_{t \ge 0}$ be a Brownian motion and consider the process X defined by

$$X_t := e^{-t} B_{e^{2t}}\,, \quad t \in \mathbb{R}.$$

(a) Show that $X_t \sim \mathcal{N}(0,1), \quad \forall t \in \mathbb{R}$.

(b) Show that the process $(X_t)_{t \in \mathbb{R}}$ is *time reversible*, i.e. $(X_t)_{t \ge 0} \stackrel{Law}{=} (X_{-t})_{t \ge 0}$.

Hint: Use the time inversion property of Brownian motion, i.e., if W is a Brownian motion, then

$$X_t := \begin{cases} 0, & \text{if } t = 0, \\ tW_{1/t}, & \text{if } t > 0, \end{cases}$$

is also a Brownian motion.

Exercise 3.3 Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables with $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ for each $n \in \mathbb{N}$.

(a) Show that if the sequence $(X_n)_{n \in \mathbb{N}}$ converges in distribution to a random variable X, then the limits $\mu := \lim_{n \to \infty} \mu_n$ and $\sigma^2 := \lim_{n \to \infty} \sigma_n^2$ exist and $X \sim \mathcal{N}(\mu, \sigma^2)$.

(b) Show that if $(X_n)_{n \in \mathbb{N}}$ is a Gaussian process indexed by $\mathbb{N}$ and converges in probability to a random variable X as n goes to infinity, then it converges also in L^2 to X.

Exercise 3.4 Matlab Exercise The goal of this exercise is illustrate the Wiener-Lévy representation of Brownian motion. Therefore, for $n \in \mathbb{N}$ let $\phi_{n,k}$ and ϕ_0 denote the Schauder functions, i.e.,

$$\begin{aligned} \phi_0(t) &:= t \\ \phi_{n,k}(t) &:= 2^{n/2}(t - (k-1)2^{-n}) I_{J_{2k-1,n+1}} - 2^{n/2}(t - k2^{-n}) I_{J_{2k,n+1}}(t), \end{aligned}$$

where $I_A(t)$ denotes the indicator function on A and

$$J_{k,n} = ((k-1)2^{-n}, k2^{-n}], \quad \text{for} \quad k = 1, \dots, 2^n.$$

That is, the graph of $\phi_{n,k}$ is a triangle over $J_{k,n}$ with its peak of height $2^{-n/2-1}$ at the middle point $(2k-1)2^{-(n+1)}$. Moreover, let Y_0 and $Y_{n,k}$ be i.i.d standard normal random variables and define for $N \leq \infty$

$$W_t^N := Y_0 \phi_0(t) + \sum_{n=0}^{N} \sum_{k=1}^{2^n} Y_{n,k} \phi_{n,k}(t).$$

We know from the lecture that W^∞ is well-defined and is a Brownian motion.
Simulate 10 sample paths of the process W^N with $N = 12$. In this exercise you can set $T = 1$ and use an equidistant time grid with 2000 grid points, i.e., $t_i = i/M, i = 0, \ldots, M = 2 \cdot 10^3$.
Hint:

- First write a function *schauderfunction(n,k,t)* which computes the schauder functions for given n, k and t
- Figure out how many iid normal random variables you need and compute W^N by sequentially adding the new increments