Brownian Motion and Stochastic Calculus

Exercise sheet 3

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than March 17th

Exercise 3.1 Let $X = (X^1, \ldots, X^d)$ be an $\mathbb{R}^d$-valued stochastic process on $[0, 1]$. For $x = (x^1, \ldots, x^d) \in \mathbb{R}^d$ with Euclidean norm $\|x\| = 1$, we define the process $Y^x = (Y^x_t)_{0 \le t \le 1}$ by

$$Y^x_t = x^\top X_t = \sum_{i=1}^d x^i X^i_t.$$

Prove that if X is a Brownian motion in $\mathbb{R}^d$, then every Y^x is a Brownian motion in $\mathbb{R}$.

Solution 3.1 It is clear that Y^x is $\mathbb{P}$-a.s. continuous, and a centred Gaussian process. Additionally, for all $s \le t$

$$\mathbb{E}\left[Y^x_s Y^x_t\right] = \mathbb{E}\left[\sum_{i,j=1}^d x_i x_j X^i_s X^j_t\right] = \sum_{i,j=1}^d x_i x_j \mathbb{E}\left[X^i_s X^j_t\right] = \sum_{i=1}^d x_i^2 \mathbb{E}\left[X^i_s X^j_t\right] = s$$

where we have used that due to the independence the cross terma are always 0. Proposition (1.4) of the script implies our result.

Exercise 3.2 Let $(B_t)_{t\geq 0}$ be a Brownian motion and consider the process X defined by

$$X_t := e^{-t} B_{e^{2t}}\,, \quad t \in \mathbb{R}.$$

(a) Show that $X_t \sim \mathcal{N}(0,1), \quad \forall t \in \mathbb{R}$.

(b) Show that the process $(X_t)_{t\in\mathbb{R}}$ is *time reversible*, i.e. $(X_t)_{t\geq 0} \stackrel{Law}{=} (X_{-t})_{t\geq 0}$.

Hint: Use the time inversion property of Brownian motion, i.e., if W is a Brownian motion, then

$$X_t := \begin{cases} 0, & \text{if } t = 0, \\ tW_{1/t}, & \text{if } t > 0, \end{cases}$$

is also a Brownian motion.

Solution 3.2

(a) Fix any $t \in \mathbb{R}$. Since Brownian motion B is a Gaussian process, we get by definition that X_t is Gaussian distributed. It remains to check its mean and variance:

$$\begin{aligned} \mathrm{E}[X_t] &= 0, \\ \mathrm{Var}(X_t) &= e^{-2t}e^{2t} = 1. \end{aligned}$$

(b) Fix any $n \in \mathbb{N}$ and any $t_1, t_2, ..., t_n \geq 0$. It is enough to check that

$$\big(X_{-t_1}, X_{-t_2}, ..., X_{-t_n}\big) \stackrel{Law}{=} \big(X_{t_1}, X_{t_2}, ..., X_{t_n}\big).$$

From the invariance by time inversion property of Brownian motion (cf. Proposition 1.1 in Section 2.1)), we get that for any $\tilde{t}_1, ..., \tilde{t}_n \geq 0$

$$\big(\tilde{t}_1 B_{1/\tilde{t}_1}, \tilde{t}_2 B_{1/\tilde{t}_2}, ..., \tilde{t}_n B_{1/\tilde{t}_n}\big) \stackrel{Law}{=} \big(B_{\tilde{t}_1}, B_{\tilde{t}_2}, ..., B_{\tilde{t}_n}\big).$$

Therefore, for $\tilde{t}_i := e^{-2t_i}$, $i := 1, ..., n$, we get that

$$\begin{aligned} \big(X_{-t_1}, X_{-t_2}, ..., X_{-t_n}\big) &= \big(e^{t_1} B_{e^{-2t_1}}, e^{t_2} B_{e^{-2t_2}}, ..., e^{t_n} B_{e^{-2t_n}}\big) \\ &\stackrel{Law}{=} \big(e^{-t_1} B_{e^{2t_1}}, e^{-t_2} B_{e^{2t_2}}, ..., e^{-t_n} B_{e^{2t_n}}\big) \\ &= \big(X_{t_1}, X_{t_2}, ..., X_{t_n}\big). \end{aligned}$$

Exercise 3.3 Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables with $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ for each $n \in \mathbb{N}$.

(a) Show that if the sequence $(X_n)_{n\in\mathbb{N}}$ converges in distribution to a random variable X, then the limits $\mu := \lim_{n\to\infty} \mu_n$ and $\sigma^2 := \lim_{n\to\infty} \sigma_n^2$ exist and $X \sim \mathcal{N}(\mu, \sigma^2)$.

(b) Show that if $(X_n)_{n\in\mathbb{N}}$ is a Gaussian process indexed by $\mathbb{N}$ and converges in probability to a random variable X as n goes to infinity, then it converges also in L^2 to X.

Solution 3.3 Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables with $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ for each $n \in \mathbb{N}$.

(a) Since $(X_n)_{n\in\mathbb{N}}$ converges in distribution to X, we know from the continuity theorem for characteristic functions that for any $t \in \mathbb{R}$

$$\varphi_{X_n}(t) = \exp\Big(it\mu_n - \frac{t^2\sigma_n^2}{2}\Big) \longrightarrow \varphi_X(t) \quad \text{as } n \to \infty. \tag{1}$$

By taking absolute values, we see that

$$|\varphi_X(t)| = \lim_{n\to\infty} \exp\Big(-\frac{t^2\sigma_n^2}{2}\Big). \tag{2}$$

Moreover, φ_X is continuous in 0. Therefore, as $\varphi_X(0) = 1$, we can find $t_0 \neq 0$ such that $\varphi(t_0) \neq 0$. Taking the logarithm in (2), we see that the $\lim_{n\to\infty} \sigma_n^2$ exists and

$$\lim_{n\to\infty} \sigma_n^2 = -\frac{2}{t_0^2} \log|\varphi_X(t_0)| =: \sigma^2$$

As a consequence, due to (1), we see that the sequence

$$\exp\big(it\mu_n\big) = \exp\Big(\frac{t^2\sigma_n^2}{2}\Big)\varphi_{X_n}(t) \tag{3}$$

converges pointwise for any $t \in \mathbb{R}$ as n goes to infinity.
Next, we prove that the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges. Set

$$\underline{\mu} := \liminf_{n\to\infty} \mu_n \quad \text{and} \quad \overline{\mu} := \limsup_{n\to\infty} \mu_n.$$

We claim that $\overline{\mu} < \infty$. Assume by contradiction that $\overline{\mu} = \infty$. In that case, we find a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ which diverge to infinity. For any point $a \in \mathbb{R}$ such that $P[X = a] = 0$, we deduce from the Portemonteau theorem of weak convergence that

$$\lim_{k\to\infty} P[X_{n_k} \le a] = P[X \le a].$$

Let $Y \sim \mathcal{N}(0, 1)$. By definition of X_{n_k},

$$P[X_{n_k} \le a] = P[\mu_{n_k} + \sigma_{n_k} Y \le a].$$

By the divergence property of the sequence $(\mu_{n_k})_{k\in\mathbb{N}}$, since $(\sigma_{n_k})_{k\in\mathbb{N}}$ converges, we conclude that $\mu_{n_k} + \sigma_{n_k} Y$ converges P-a.s. to infinity. Thus, we get that $P[X \le a] = 0$. But since we can find arbitrarily big points a satisfying $P[X = a] = 0$, we get a contradiction to the fact that $\lim_{a\to\infty} P[X \le a] = 1$ by the definition of a cumulative distribution function. Thus, we conclude that $\overline{\mu} < \infty$. With a similar argument, one can show that $\underline{\mu} > -\infty$. Therefore, we deduce from the pointwise convergence of the sequence in (3) that for any $t \in \mathbb{R}$

$$\exp\big(it\underline{\mu}\big) = \exp\big(it\overline{\mu}\big).$$

Thus, we get that for any $t \in \mathbb{R}$

$$t(\overline{\mu} - \underline{\mu}) \equiv 0 \pmod{2\pi}$$

which implies that $\underline{\mu} = \overline{\mu}$. In other words, $\mu := \lim_{n\to\infty} \mu_n$ exists. As a consequence of (1), we get that for any $t \in \mathbb{R}$

$$\varphi_X(t) = \exp\left(it\mu - \frac{t^2\sigma^2}{2}\right)$$

and thus, $X \sim \mathcal{N}(\mu, \sigma^2)$.

(b) Since $(X_n)_{n\in\mathbb{N}}$ converges in probability to X, $(X_n - X)_{n\in\mathbb{N}}$ converges in probability to 0 and hence $(X_n - X)_{n\in\mathbb{N}}$ converges in distribution to 0.
Fix any $n \in \mathbb{N}$. The sequence $(X_n - X_k)_{k\in\mathbb{N}}$ converges in probability to $X_n - X$ and hence $(X_n - X_k)_{k\in\mathbb{N}}$ converges in distribution to $X_n - X$. Now, since by assumption $(X_n)_{n\in\mathbb{N}}$ is a Gaussian process, we get that for each k, $X_n - X_k$ is normal distributed. Thus, we deduce from part a) that $X_n - X$ is normal distributed. Since $n \in \mathbb{N}$ was arbitrarily chosen, we get that $(X_n - X)_{n\in\mathbb{N}}$ is a sequence of Gaussian random variables. Moreover, since $(X_n - X)_{n\in\mathbb{N}}$ converges in distribution to 0, we deduce again from a) that

$$E[X_n - X] \longrightarrow 0 \quad \text{and} \quad \operatorname{Var}(X_n - X) \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

As a consequence, we get directly the L^2 convergence of X_n to X, since

$$\|X_n - X\|_{L^2}^2 = E\big[|X_n - X|^2\big] = \big(E[X_n - X]\big)^2 + \operatorname{Var}\big(X_n - X\big).$$

Exercise 3.4 Matlab Exercise The goal of this exercise is illustrate the Wiener-Lévy representation of Brownian motion. Therefore, for $n \in \mathbb{N}$ let $\phi_{n,k}$ and ϕ_0 denote the Schauder functions, i.e.,

$$\begin{aligned}\phi_0(t) &:= t \\ \phi_{n,k}(t) &:= 2^{n/2}(t-(k-1)2^{-n})I_{J_{2k-1,n+1}} - 2^{n/2}(t-k2^{-n})I_{J_{2k,n+1}}(t),\end{aligned}$$

where $I_A(t)$ denotes the indicator function on A and

$$J_{k,n} = ((k-1)2^{-n}, k2^{-n}], \quad \text{for} \quad k = 1,\dots,2^n.$$

That is, the graph of $\phi_{n,k}$ is a triangle over $J_{k,n}$ with its peak of height $2^{-n/2-1}$ at the middle point $(2k-1)2^{-(n+1)}$. Moreover, let Y_0 and $Y_{n,k}$ be i.i.d standard normal random variables and define for $N \leq \infty$

$$W_t^N := Y_0\phi_0(t) + \sum_{n=0}^{N}\sum_{k=1}^{2^n} Y_{n,k}\phi_{n,k}(t).$$

We know from the lecture that W^∞ is well-defined and is a Brownian motion.
Simulate 10 sample paths of the process W^N with $N = 12$. In this exercise you can set $T = 1$ and use an equidistant time grid with 2000 grid points, i.e., $t_i = i/M, i = 0,\dots,M = 2\cdot 10^3$.
Hint:

- First write a function *schauderfunction(n,k,t)* which computes the schauder functions for given n, k and t
- Figure out how many iid normal random variables you need and compute W^N by sequentially adding the new increments

Solution 3.4 Matlab Files

```
function bmscex34
% In this exercise we simulate Brownian motion using the Wiener-Levy
% Representation (see Corollary I.(5.16) in the lecture notes)

% upper bound on n
nmax=12;
% number of iid normal variables
N=sum(2.^[1:nmax]);
% number of sample paths
M=10;
% final time
T=1;
% number of grid points
gridpoi=2000;
% time grid
grid=0:T/gridpoi:T;
% iid std normal random variables
Y=randn(N,M);
% output matrix (N,M)=(N*1)*(1*M) matrix, initialize for n=0: Y_0*phi_0(t)
out=grid'*randn(1,M);

% use the definition of W^N
for n=1:nmax
    for k=1:(2^n)
```

```
        % formula I.(5.8)
    out=out+(schauderba(n,k,grid))'*Y(2^(n-1)+k,:);
    end
end
plot(grid,out)
title('BM with Wiener-Levy representation');
xlabel('time');
ylabel('value');
end


function[value]=schauderba(n,k,t)
% the function implements the schauderbasis function see definition I
    (5.7)
ind1=t> (2*k-2)*2^(-(n+1));
ind2=t<= (2*k-1)*2^(-(n+1));

ind3=1-ind2;
ind4=t<=2*k*2^(-(n+1));

% Definition of the Schauder basis function definition I (5.7)
value=(ind1.*ind2).*2^(n/2).*(t-(k-1)*2^(-n))...
    -(ind3.*ind4).*2^(n/2).*(t-k*2^(-n));
end
```