Coordinator
A. Sepúlveda

Brownian Motion and Stochastic Calculus

Exercise sheet 4

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than March 24th

Exercise 4.1 Let W be a Brownian motion on $[0,\infty)$ and $S_0 > 0$, $\sigma > 0$, $\mu \in \mathbb{R}$ constants. The stochastic process $S = (S_t)_{t\geq 0}$ given by

$$S_t = S_0 \exp\big(\sigma W_t + (\mu - \sigma^2/2)t\big)$$

is called *geometric Brownian motion.*

(a) Prove that for $\mu \neq \sigma^2/2$, we have

$$\lim_{t\to\infty} S_t = +\infty \quad P\text{-a.s.} \quad \text{or} \quad \lim_{t\to\infty} S_t = 0 \quad P\text{-a.s.}$$

When do the respective cases arise?

(b) Discuss the behaviour of S_t as $t \to \infty$ in the case $\mu = \sigma^2/2$.

(c) For $\mu = 0$, show that S is a martingale, but not uniformly integrable.

Exercise 4.2 Consider two stopping times σ, τ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. The goal of this exercise is to show that

$$E[E[\,\cdot\,|\mathcal{F}_\sigma]|\mathcal{F}_\tau] = E[\,\cdot\,|\mathcal{F}_{\sigma\wedge\tau}] = E[E[\,\cdot\,|\mathcal{F}_\tau]|\mathcal{F}_\sigma] \quad P\text{-a.s.,} \tag{$\star$}$$

i.e., the operators $E[\,\cdot\,|\mathcal{F}_\tau]$ and $E[\,\cdot\,|\mathcal{F}_\sigma]$ commute and their composition equals $E[\,\cdot\,|\mathcal{F}_{\sigma\wedge\tau}]$.
Remark: For arbitrary sub-σ-algebras $\mathcal{G}, \mathcal{G}' \subset \mathcal{F}$, the conditional expectations $E[E[\,\cdot\,|\mathcal{G}]|\mathcal{G}']$, $E[E[\,\cdot\,|\mathcal{G}']|\mathcal{G}]$ and $E[\,\cdot\,|\mathcal{G}\cap\mathcal{G}']$ do **not** coincide in general.

(a) Show that $\sigma \wedge \tau$ is a stopping time and $\mathcal{F}_{\sigma\wedge\tau} = \mathcal{F}_\tau \cap \mathcal{F}_\sigma$.

(b) Show that if $A \in \mathcal{F}_\sigma$, then $A \cap \{\sigma \leq \tau\}$ and $A \cap \{\sigma < \tau\}$ belong to $\mathcal{F}_\tau$.
Hint: For the second assertion, use that $a < b$ if and only if there is a rational q such that $a \leq q < b$.

(c) Conclude that $\{\sigma \leq \tau\}, \{\sigma < \tau\} \in \mathcal{F}_{\sigma\wedge\tau}$.

(d) Show that $E[Y|\mathcal{F}_\tau]$ is $\mathcal{F}_{\sigma\wedge\tau}$-measurable if Y is an integrable $\mathcal{F}_\sigma$-measurable random variable. Conclude ($\star$).

(e) Let $M = (M_t)_{t\geq 0}$ be a right-continuous martingale. Show that the stopped process $M^\tau = (M_{\tau\wedge t})_{t\geq 0}$ is again a martingale.
Hint: Use ($\star$) and the stopping theorem.

Exercise 4.3 Let $(\Omega, \mathcal{F}, \mathcal{F}_t)$ a filtered probability space. Take N a continuous positive martingale (i.e. almost surely $N_t \geq 0$) with $N_0 = 1$ and $N_t \to 0$ as $t \to \infty$.

(a) Show one example of a martingale satisfying this conditions.

(b) Show that for all $a > 1$, $T_a := \inf\{t \geq 0, N_t = a\}$ is a stopping time.

(c) Use the stopping time theorem to show that $\sup_{t\geq 0} N_t \overset{law}{=} 1/U$, where U is a uniform random variable.
Hint: It may be useful to note that $\{\sup_{t\geq 0} N_t \geq a\} = \{T_a < \infty\}$.