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Brownian Motion and Stochastic Calculus

Exercise sheet 4

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than March 24th

Exercise 4.1 Let W be a Brownian motion on $[0,\infty)$ and $S_0 > 0$, $\sigma > 0$, $\mu \in \mathbb{R}$ constants. The stochastic process $S = (S_t)_{t\geq 0}$ given by

$$S_t = S_0 \exp\big(\sigma W_t + (\mu - \sigma^2/2)t\big)$$

is called *geometric Brownian motion.*

(a) Prove that for $\mu \neq \sigma^2/2$, we have

$$\lim_{t\to\infty} S_t = +\infty \quad P\text{-a.s.} \quad \text{or} \quad \lim_{t\to\infty} S_t = 0 \quad P\text{-a.s.}$$

When do the respective cases arise?

(b) Discuss the behaviour of S_t as $t \to \infty$ in the case $\mu = \sigma^2/2$.

(c) For $\mu = 0$, show that S is a martingale, but not uniformly integrable.

Solution 4.1

(a) Noting that a.s. $W_t/t \to 0$ we have that:

- If $(\mu - \sigma^2/2) > 0$ a.s. $\sigma W_t + (\mu - \sigma^2/2) \to \infty$, thus $\lim S_t = \infty$.
- If $(\mu - \sigma^2/2) < 0$ a.s. $\sigma W_t + (\mu - \sigma^2/2) \to -\infty$, thus $\lim S_t = 0$.

(b) The fact that a.s. $\liminf B_t = -\infty$ and $\limsup B_t = \infty$ implies that when $\mu = \sigma^2/2$, $\liminf S_t = 0$ and $\limsup S_t = \infty$.

(c) Note that if $s \leq t$ we have that $W_t - W_s$ is independent of $\mathcal{F}_s$ and follows the law of a centred normal with variance $t - s$

$$\begin{aligned}\mathbb{E}\left[S_t \mid \mathcal{F}_s\right] &= S_0 \mathbb{E}\left[\exp(\sigma(W_t - W_s) + \sigma W_s - \sigma^2 t/2) \mid \mathcal{F}_s\right] \\ &= S_0 \exp(\sigma W_s - \sigma^2 s/2)\mathbb{E}\left[\exp(\sigma(W_t - W_s) - \sigma^2(t-s)/2\right] = S_s.\end{aligned}$$

Thus S_t is a martingale that converges to 0 a.s.. To see it is not uniformly integrable suppose it is. We would have that $S_0 = \mathbb{E}\left[\lim_{t\to 0} S_t\right] = 0$.

Exercise 4.2 Consider two stopping times σ, τ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. The goal of this exercise is to show that

$$E[E[\,\cdot\,|\mathcal{F}_\sigma]|\mathcal{F}_\tau] = E[\,\cdot\,|\mathcal{F}_{\sigma\wedge\tau}] = E[E[\,\cdot\,|\mathcal{F}_\tau]|\mathcal{F}_\sigma] \quad P\text{-a.s.}, \tag{$\star$}$$

i.e., the operators $E[\,\cdot\,|\mathcal{F}_\tau]$ and $E[\,\cdot\,|\mathcal{F}_\sigma]$ commute and their composition equals $E[\,\cdot\,|\mathcal{F}_{\sigma\wedge\tau}]$.
Remark: For arbitrary sub-σ-algebras $\mathcal{G}, \mathcal{G}' \subset \mathcal{F}$, the conditional expectations $E[E[\,\cdot\,|\mathcal{G}]|\mathcal{G}']$, $E[E[\,\cdot\,|\mathcal{G}']|\mathcal{G}]$ and $E[\,\cdot\,|\mathcal{G}\cap\mathcal{G}']$ do **not** coincide in general.

(a) Show that $\sigma\wedge\tau$ is a stopping time and $\mathcal{F}_{\sigma\wedge\tau} = \mathcal{F}_\tau \cap \mathcal{F}_\sigma$.

(b) Show that if $A \in \mathcal{F}_\sigma$, then $A \cap \{\sigma \le \tau\}$ and $A \cap \{\sigma < \tau\}$ belong to $\mathcal{F}_\tau$.
Hint: For the second assertion, use that $a < b$ if and only if there is a rational q such that $a \le q < b$.

(c) Conclude that $\{\sigma \le \tau\}, \{\sigma < \tau\} \in \mathcal{F}_{\sigma\wedge\tau}$.

(d) Show that $E[Y|\mathcal{F}_\tau]$ is $\mathcal{F}_{\sigma\wedge\tau}$-measurable if Y is an integrable $\mathcal{F}_\sigma$-measurable random variable. Conclude ($\star$).

(e) Let $M = (M_t)_{t\ge0}$ be a right-continuous martingale. Show that the stopped process $M^\tau = (M_{\tau\wedge t})_{t\ge0}$ is again a martingale.
Hint: Use ($\star$) and the stopping theorem.

Solution 4.2

(a) Note that $\{\sigma\wedge\tau \le t\} = \{\sigma \le t\} \cup \{\tau \le t\}$, because each of this element belongs to $\mathcal{F}_t$ we have that $\sigma\wedge\tau$ is a stopping time.

For the second part let us firt prove that $\mathcal{F}_{\sigma\wedge\tau} \subseteq \mathcal{F}_\sigma \cap \mathcal{F}_\sigma$. Take $A \in \mathcal{F}_{\sigma\wedge\tau}$, by definitioni $(A \cap \{\sigma\wedge\tau \le t\} \in \mathcal{F}_t$ from where it follows that,

$$A \cap \{\sigma \le t\} = (A \cap \{\sigma\wedge\tau \le t\}) \cap (\{\sigma \le t\}) \in \mathcal{F}_t.$$

Thus $\mathcal{F}_{\sigma\wedge\tau} \subseteq \mathcal{F}_s$, doing the same thing for τ we have that $\mathcal{F}_{\sigma\wedge\tau} \subseteq \mathcal{F}_s \cap \mathcal{F}_t$. For the other inclusion, let $A \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau$. We see that

$$A \cap \{\sigma\wedge\tau \le t\} = A \cap \Big(\{\sigma \le t\} \cup \{\tau \le t\}\Big) = \Big(A \cap \{\sigma \le t\}\Big) \cup \Big(A \cap \{\tau \le t\}\Big) \in \mathcal{F}_t$$

and thus, $A \in \mathcal{F}_{\sigma\wedge\tau}$.

(b) Take $A \in \mathcal{F}_\sigma$. To see that $A \cap \{\sigma < \tau\} \in \mathcal{F}_\tau$ note that

$$(A \cap \{\sigma < \tau\}) \cap \{\tau \le t\} = (A \cap \{\tau \le t\}) \cap \bigcup_{q\in[0,t]\cap\mathbb{Q}} \{\sigma \le q\} \cap \{q < \tau\} \in \mathcal{F}_t.$$

To see that $A \cap \{\sigma \le \tau\} \in \mathcal{F}_\tau$, note that thanks to (a) for any $t \ge 0$, $\sigma\wedge t$ is a stopping time and $\mathcal{F}_{\sigma\wedge t} \subseteq \mathcal{F}_t$, thus $\sigma\wedge t$ and $\tau\wedge t$ are $\mathcal{F}_t$ measurable. Then for any $t \ge 0$,

$$A \cap \{\sigma \le \tau\} \cap \{\tau \le t\} = (A \cap \{\tau \le t\}) \cap \{\sigma\wedge t \le \tau\wedge t\} \in \mathcal{F}_t.$$

This is because the first and the second term belong to $\mathcal{F}_t$.

(c) Let $t \ge 0$. By taking $A = \Omega$ in part (b) we know that $\{\sigma \le \tau\}, \{\sigma < \tau\} \in \mathcal{F}_\tau$ and $\{\tau \le \sigma\}, \{\tau < \sigma\} \in \mathcal{F}_\sigma$. Thus by taking complements $\{\sigma \le \tau\}, \{\sigma < \tau\} \in \mathcal{F}_\tau \cap \mathcal{F}_\sigma = \mathcal{F}_{\sigma\wedge\tau}$.

(d) Let us first prove that if Y is $\mathcal{F}_\sigma$-measurable then $Y\mathbf{1}_{\{\sigma\leq\tau\}}$ is $\mathcal{F}_{\tau\wedge\sigma}$ measurable. Due to the fact that $\{\sigma\leq\tau\}\in\mathcal{F}_\sigma$ we have that $Y\mathbf{1}_{\{\sigma\leq\tau\}}$ is $\mathcal{F}_\sigma$ measurable. Now let us prove that it is $\mathcal{F}_\tau$-measurable. To do this note that thanks to problem (b) this is true for any step function, i.e., if $A_1,...,A_n\in\mathcal{F}_\sigma$ and $\lambda_1,...,\lambda_n\in\mathbb{R}$ then taking $Y=\sum_{i=1}^n\lambda_i\mathbf{1}_{A_i}$ we have that $Y\mathbf{1}_{\{\sigma\leq\tau\}}$ is $\mathcal{F}_\tau$-measurable. For a general Y, we take $Y^{(n)}$ a sequence of step function converging to Y, we have that $Y^{(n)}\mathbf{1}_{\{\sigma\leq\tau\}}\to Y\mathbf{1}_{\{\sigma\leq\tau\}}$. Thus, $Y\mathbf{1}_{\{\sigma\leq\tau\}}$ is $\mathcal{F}_\tau$-measurable. We also have that if Z is $\mathcal{F}_\tau$-measurable, $Z\mathbf{1}_{\{\tau<\sigma\}}$ is $\mathcal{F}_{\sigma\wedge\tau}$-measurable.

Now, let us see that

$$\mathbb{E}\left[Y\mid\mathcal{F}_\tau\right]=\mathbb{E}\left[Y\mathbf{1}_{\{\tau<\sigma\}}\mid\mathcal{F}_\tau\right]+\mathbb{E}\left[Y\mathbf{1}_{\{\sigma\leq\tau\}}\mid\mathcal{F}_\tau\right]=\mathbb{E}\left[Y\mid\mathcal{F}_\tau\right]\mathbf{1}_{\{\tau<\sigma\}}+Y\mathbf{1}_{\{\sigma\leq\tau\}}$$

each term is $\mathcal{F}_{\sigma\wedge\tau}$-measurable from where we conclude that $\mathbb{E}\left[Y\mid\mathcal{F}_\tau\right]$ is $\mathcal{F}_{\sigma\wedge\tau}$-measurable.

To see ($\star$) it is just enough to note that for any integrable r.v. Z, $\mathbb{E}\left[\mathbb{E}\left[Z\mid\mathcal{F}_\sigma\right]\mid\mathcal{F}_\tau\right]$ is $\mathcal{F}_{\sigma\wedge\tau}$-measurable, thus it is equal to $\mathbb{E}\left[\mathbb{E}\left[Z\mid\mathcal{F}_\sigma\right]\mid\mathcal{F}_{\tau\wedge\sigma}\right]=\mathbb{E}\left[Z\mid\mathcal{F}_{\sigma\wedge\tau}\right]$, thanks to the tower property.

(e) Take $s\geq t$ and note that $\tau\wedge s\leq\tau\wedge t$ are bounded stopping times, then thanks to the Stopping theorem

$$\mathbb{E}\left[M_{\tau\wedge t}\mid\mathcal{F}_s\right]=\mathbb{E}\left[\mathbb{E}\left[M_t\mid\mathcal{F}_{\tau\wedge t}\right]\mid\mathcal{F}_s\right]=\mathbb{E}\left[M_t\mid\mathcal{F}_{s\wedge\tau}\right]=M_{\tau\wedge s}$$

where in the second equality we used ($\star$).

Exercise 4.3 Let $(\Omega, \mathcal{F}, \mathcal{F}_t)$ a filtered probability space. Take N a continuous positive martingale (i.e. almost surely $N_t \geq 0$) with $N_0 = 1$ and $N_t \to 0$ as $t \to \infty$.

(a) Show one example of a martingale satisfying this conditions.

(b) Show that for all $a > 1$, $T_a := \inf\{t \geq 0, N_t = a\}$ is a stopping time.

(c) Use the stopping time theorem to show that $\sup_{t \geq 0} N_t \overset{law}{=} 1/U$, where U is a uniform random variable.
Hint: It may be useful to note that $\{\sup_{t \geq 0} N_t \geq a\} = \{T_a < \infty\}$.

Solution 4.3

(a) Take $\mu = 0$ and use S_t of Exercise 4.1.

(b) Note that because N is continuous and $N_0 = 1$

$$\{T_a \leq t\} = \bigcap_{\epsilon \in (0,1) \cap \mathbb{Q}} \bigcup_{q \in (0,t) \cap \mathbb{Q}} \{N_q \geq a - \epsilon\} \in \mathcal{F}_t.$$

(c) We have that $N_{t \wedge T_a}$ is a bounded martingale (see exercise 2. (e)) thus bounded convergence theorem

$$1 = \mathbb{E}\left[N_t \wedge T_a\right] \overset{t \to \infty}{\to} \mathbb{E}\left[N_{T_a} \mathbf{1}_{\{T_a < \infty\}}\right] = a\mathbb{P}(T_a < \infty).$$

Thus, thanks to the hint $\mathbb{P}(\sup N_t \geq a) = 1/a$, i.e., $\mathbb{P}(1/\sup N_t \leq a) = a$ which is exactly the law of a uniform random variable.