ETHZ, Spring 2017
D-MATH
Prof. Dr. Martin Larsson

Coordinator
A. Sepúlveda

Brownian Motion and Stochastic Calculus

Exercise sheet 5

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than March 31th

Exercise 5.1 Let $(W_t)_{t\geq 0}$ be a Brownian motion. Moreover, let $\mathbb{F}^0$ be the (raw) filtration generated by W and $\mathbb{F}$ its right-continuous modification. Let τ be an $\mathbb{F}$-stopping time with $\tau < \infty$ P-a.s. and $\widetilde{W}_\cdot := W_{\tau+\cdot} - W_\tau$. Prove that $\widetilde{W}$ is independent of $\mathcal{F}_\tau$.
Hint: Use the monotone class theorem and approximate τ from above as on page 46 of the script.

Exercise 5.2 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$ and define the *integrated Brownian motion* $Y = (Y_t)_{t\geq 0}$ by $Y_t = \int_0^t W_s ds$. Moreover, let $\mathbb{F}^W := (\mathcal{F}_t^W)_{t\geq 0}$ be the raw filtration generated by W.

(a) For each $h \geq 0$, show that the pair (W_h, Y_h) has a two-dimensional normal distribution with mean zero and covariance matrix given by

$$\begin{pmatrix} h & h^2/2 \\ h^2/2 & h^3/3 \end{pmatrix}.$$

Hint: First, show that (W_h, Y_h) has a two-dimensional normal distribution by approximating Y_h by Riemann sums and using Exercise **3-3**. Second, use Fubini's theorem to compute the covariance matrix.

(b) Show that the pair (W, Y) is a (homogeneous) Markov process with state space $\mathbb{R}^2$, filtration $\mathbb{F}^W$ and transition semigroup $(K_h)_{h\geq 0}$ given by

$$K_h\big((w,y),\cdot\big) = \mathcal{N}\left(\begin{pmatrix} w \\ y + hw \end{pmatrix}, \begin{pmatrix} h & h^2/2 \\ h^2/2 & h^3/3 \end{pmatrix} \right), \quad h \geq 0.$$

(c) Show that Y *alone* is **not** a Markov process with respect to $\mathbb{F}^W$.

Exercise 5.3

(a) Prove that P-almost all Brownian paths are nowhere on $[0,1]$ Hölder-continuous of order α, for any $\alpha > \frac{1}{2}$.
Hint: Take any $M \in \mathbb{N}$ satisfying $M(\alpha - \frac{1}{2}) > 1$ and show that the set $\{B_\cdot(\omega)$ is α-Hölder at some $s \in [0,1]\}$ is contained in the set $\bigcup_{C\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} \bigcup_{k=0,\dots,n-1} \bigcap_{j=1}^{M} \left\{ |B_{\frac{k+j}{n}}(\omega) - B_{\frac{k+j-1}{n}}(\omega)| \leq C \frac{1}{n^\alpha} \right\}$.

(b) The *Kolmogorov-Čentsov theorem* states that a process X on $[0,T]$ satisfying

$$E\big[|X_t - X_s|^\alpha\big] \leq C\,|t-s|^{1+\beta}, \quad s,t \in [0,T],$$

where $\alpha, \beta, C > 0$, has a version which is locally Hölder-continuous of order γ for all $\gamma < \beta/\alpha$. Use this to deduce that Brownian motion has for every $\gamma < 1/2$ a version which is locally Hölder-continuous of order γ.

Remark: One can also show that the Brownian paths are *not* Hölder-continuous of order 1/2. The exact modulus of continuity was found by P. Lévy.