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Brownian Motion and Stochastic Calculus

Exercise sheet 5

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than March 31th

Exercise 5.1 Let $(W_t)_{t\geq 0}$ be a Brownian motion. Moreover, let $\mathbb{F}^0$ be the (raw) filtration generated by W and $\mathbb{F}$ its right-continuous modification. Let τ be an $\mathbb{F}$-stopping time with $\tau < \infty$ P-a.s. and $\widetilde{W}_\cdot := W_{\tau+\cdot} - W_\tau$. Prove that $\widetilde{W}$ is independent of $\mathcal{F}_\tau$.
Hint: Use the monotone class theorem and approximate τ from above as on page 46 of the script.

Solution 5.1 We have to prove the independence of the σ-fields $\sigma(\widetilde{W}_t; t \geq 0)$ and $\mathcal{F}_\tau$. By Dynkin's theorem (and since $\widetilde{W}_0 = 0$ P-a.s.), it is enough to prove the independence of $\mathcal{F}_\tau$ and the family $\mathcal{Z}$ of cylinder sets

$$\mathcal{Z} = \Big\{ \{\widetilde{W}_{t_1} \in A_1, \widetilde{W}_{t_2} - \widetilde{W}_{t_1} \in A_2, \dots, \widetilde{W}_{t_n} - \widetilde{W}_{t_{n-1}} \in A_n\}; \\ n \in \mathbb{N},\ 0 \leq t_1 \leq \dots \leq t_n,\ A_1, \dots A_n \in \mathcal{B}(\mathbb{R}) \Big\},$$

as $\mathcal{Z}$ is a π-system generating $\sigma(\widetilde{W}_t; t \geq 0)$. As usual, approximate τ with the sequence of stopping times $\tau_m = \sum_{k\geq 0} \frac{k+1}{2^m} \mathbf{1}_{\{\frac{k}{2^m} \leq \tau < \frac{k+1}{2^m}\}}$, and define the processes $\widetilde{W}^m_\cdot = W_{\tau_m+\cdot} - W_{\tau_m}$. Take $A \in \mathcal{F}_\tau$ and $f_1, \dots f_n$ bounded and continuous on $\mathbb{R}$. As τ_m only takes a countable number of different values, we get for $t_0 = 0$

$$\begin{aligned} &E\Big[\mathbf{1}_A \prod_{i=1}^n f_i(\widetilde{W}^m_{t_i} - \widetilde{W}^m_{t_{i-1}})\Big] \\ &= \sum_{k\geq 0} E\Big[\mathbf{1}_{\{\frac{k}{2^m} \leq \tau < \frac{k+1}{2^m}\}} \mathbf{1}_A \prod_{i=1}^n f_i(W_{\frac{k+1}{2^m}+t_i} - W_{\frac{k+1}{2^m}+t_{i-1}})\Big] \\ &= \sum_{k\geq 0} E\Big[\prod_{i=1}^n f_i(W_{\frac{k+1}{2^m}+t_i} - W_{\frac{k+1}{2^m}+t_{i-1}})\Big] P\Big[\Big\{\frac{k}{2^m} \leq \tau < \frac{k+1}{2^m}\Big\} \cap A\Big], \end{aligned}$$

since $\{\frac{k}{2^m} \leq \tau < \frac{k+1}{2^m}\} \cap A$ is in $\mathcal{F}_{\frac{k+1}{2^m}}$ and $W_{\frac{k+1}{2^m}+\cdot} - W_{\frac{k+1}{2^m}}$ is independent of $\mathcal{F}_{\frac{k+1}{2^m}}$. By the stationarity of the Brownian increments, for each $n \in \mathbb{N}$ and $0 \leq t_0 \leq \dots \leq t_n < \infty$ we have for each $h > 0$ that $W_{t_j+h} - W_{t_{j-1}+h} \overset{(d)}{=} W_{t_j} - W_{t_{j-1}}$ for $j = 1, \dots, n$. So we get that for each $k \in \mathbb{N}_0$

$$E\Big[\prod_{i=1}^n f_i(W_{\frac{k+1}{2^m}+t_i} - W_{\frac{k+1}{2^m}+t_{i-1}})\Big] = E\Big[\prod_{i=1}^n f_i(W_{t_i} - W_{t_{i-1}})\Big].$$

Therefore with the same arguments as above for $A = \Omega$ that

$$E\Big[\prod_{i=1}^n f_i(W_{\tau_m+t_i} - W_{\tau_m+t_{i-1}})\Big] = E\Big[\prod_{i=1}^n f_i(W_{t_i} - W_{t_{i-1}})\Big].$$

This implies together with the result above that

$$E\Big[\mathbf{1}_A \prod_{i=1}^n f_i(\widetilde{W}^m_{t_i} - \widetilde{W}^m_{t_{i-1}})\Big] = E\Big[\prod_{i=1}^n f_i(W_{t_i} - W_{t_{i-1}})\Big] P[A].$$

As $m \to \infty$, $\tau_m \searrow \tau$ and $f_j(W_{\tau_m+t_j} - W_{\tau_m+t_{j-1}}) \to f_j(W_{\tau+t_j} - W_{\tau+t_{j-1}})$ P-a.s. for $j = 1, \dots, n$ by the P-a.s. right-continuity of Brownian paths and the continuity of f_j. Therefore we get with Lebesgue's dominated convergence theorem

$$E\Big[\mathbf{1}_A \prod_{i=1}^{n} f_i(\widetilde{W}_{t_i} - \widetilde{W}_{t_{i-1}})\Big] = E\Big[\prod_{i=1}^{n} f_i(\widetilde{W}_{t_i} - \widetilde{W}_{t_{i-1}})\Big] P\big[A\big]. \tag{1}$$

To extend (1) to measurable and bounded functions $f_1, \dots, f_n$ we extend it successively for each f_j with $j \in \{1, \dots, n\}$ by applying the monotone class theorem. Fix $j \in \{1, \dots, n\}$ and assume that (1) holds for $f_1, \dots, f_{j-1}$ measurable and bounded and $f_j, \dots, f_n \in C_b(\mathbb{R})$. Set $\mathcal{M} = C_b(\mathbb{R})$ and

$$\mathcal{H} = \big\{ f \text{ measurable and bounded } \big| (1) \text{ holds for } f_1, \dots, f_{j-1} \text{ measurable and bounded,} \\ f_j = f, \text{ and } f_{j+1}, \dots, f_n \in C_b(\mathbb{R}) \big\}.$$

As $C_b(\mathbb{R})$ is a vector lattice that generates $\mathcal{B}(\mathbb{R})$ (see script "Wahrscheinlichkeitstheorie", Beispiel V.1.1) (1) follows for f_j measurable and bounded by noting that $\mathcal{H}$ is a real vector space of bounded real-valued functions containing $\mathcal{M}$ and 1 and is closed under bounded monotone convergence. By plugging in indicator functions of Borel sets we finally get the independence of $\widetilde{W}$ of $\mathcal{F}_\tau$.

Exercise 5.2 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$ and define the *integrated Brownian motion* $Y = (Y_t)_{t\geq 0}$ by $Y_t = \int_0^t W_s ds$. Moreover, let $\mathbb{F}^W := (\mathcal{F}_t^W)_{t\geq 0}$ be the raw filtration generated by W.

(a) For each $h \geq 0$, show that the pair (W_h, Y_h) has a two-dimensional normal distribution with mean zero and covariance matrix given by

$$\begin{pmatrix} h & h^2/2 \\ h^2/2 & h^3/3 \end{pmatrix}.$$

Hint: First, show that (W_h, Y_h) has a two-dimensional normal distribution by approximating Y_h by Riemann sums and using Exercise **3-3**. Second, use Fubini's theorem to compute the covariance matrix.

(b) Show that the pair (W, Y) is a (homogeneous) Markov process with state space $\mathbb{R}^2$, filtration $\mathbb{F}^W$ and transition semigroup $(K_h)_{h\geq 0}$ given by

$$K_h\big((w,y),\cdot\big) = \mathcal{N}\left(\begin{pmatrix} w \\ y + hw \end{pmatrix}, \begin{pmatrix} h & h^2/2 \\ h^2/2 & h^3/3 \end{pmatrix} \right), \quad h \geq 0.$$

(c) Show that Y *alone* is **not** a Markov process with respect to $\mathbb{F}^W$.

Solution 5.2

(a) Fix $h \geq 0$. Write $(W_h, Y_h) = \lim_{n\to\infty}(W_h, \frac{h}{n}\sum_{i=0}^{n-1} W_{\frac{i}{n}h}) =: \lim_{n\to\infty}(W_h, Y_h^n)$. For each n, the random pair $(W_h, \frac{h}{n}\sum_{i=0}^{n-1} W_{\frac{i}{n}h})$ is a linear transformation of the Gaussian random vector $(W_{\frac{i}{n}h})_{i=0,\dots,n}$ and as such Gaussian itself. To show that (W_h, Y_h) is a Gaussian vector, we need to show that for any scalar $a_1, a_2 \in \mathbb{R}$, we have that $a_1 W_h + a_2 Y_h$ is normal distributed. Fix $a_1, a_2 \in \mathbb{R}$. It is clear that $a_1 W_h + a_2 Y_h = \lim_{n\to\infty} a_1 W_h + a_2 Y_h^n$ pointwise, in particular in distribution. By Exercise **3-3**, we know that the limit (in distribution, say) of Gaussian random variables is Gaussian. Thus, we conclude that $a_1 W_h + a_2 Y_h$ is a Gaussian random variable, and hence (W_h, Y_h) is a Gaussian vector. Thus, it only remains to compute mean vector and covariance matrix of (W_h, Y_h).

Clearly, $E[W_h] = 0$ and using Fubini's theorem, we see that $E[Y_h] = 0$ as well. Next, $\mathrm{Var}(W_h) = h$, and using Fubini's theorem, we get

$$\begin{aligned}\mathrm{Var}[Y_h] &= E\left[\left(\int_0^h W_r\,dr\right)^2\right] = E\left[\int_0^h W_r\,dr \int_0^h W_s\,ds\right] = E\left[\int_0^h\int_0^h W_r W_s\,dr\,ds\right] \\ &= \int_0^h\int_0^h E[W_r W_s]\,dr\,ds = \int_0^h\int_0^h r\wedge s\,dr\,ds = \int_0^h \left(\int_0^s r\,dr + \int_s^h s\,dr\right) ds \\ &= \int_0^h \left(\frac{s^2}{2} + s(h-s)\right) ds = -\frac{h^3}{6} + \frac{h^3}{2} = \frac{h^3}{3}.\end{aligned}$$

Finally, again using Fubini's theorem,

$$\mathrm{Cov}(W_h, Y_h) = E\left[W_h \int_0^h W_s\,ds\right] = \int_0^h E[W_h W_s]\,ds = \int_0^h s\,ds = \frac{h^2}{2}.$$

(b) Let $(\mathcal{F}_t^W)_{t\geq 0}$ denote the (raw) filtration generated by W, $f : \mathbb{R}^2 \to [0,\infty)$ a bounded Borel function, and $t \geq 0$, $h > 0$. By construction, Y is $(\mathcal{F}_t^W)$-adapted. Moreover, writing

$W_{t+h} = W_t + (W_{t+h} - W_t)$, $Y_{t+h} = Y_t + \int_t^{t+h} W_s\,ds = Y_t + hW_t + \int_t^{t+h}(W_s - W_t)\,ds$, and using the fact that $(W_{t+s} - W_t)_{s\geq 0}$ is independent of $\mathcal{F}_t^W$, we obtain

$$E[f(W_{t+h}, Y_{t+h})|\mathcal{F}_t^W] = E\left[f\left(w + (W_{t+h} - W_t), y + hw + \int_t^{t+h}(W_s - W_t)\,ds\right)\right]\bigg|_{w=W_t, y=Y_t}.$$

By translation invariance of Brownian motion,

$$\left(W_{t+h} - W_t, \int_t^{t+h}(W_s - W_t)\,ds\right) \stackrel{d}{=} \left(W_h, \int_0^h W_s\,ds\right) = (W_h, Y_h).$$

Thus, by part a),

$$E[f(W_{t+h}, Y_{t+h})|\mathcal{F}_t^W] = \int_{\mathbb{R}^2} f(x) K_h((W_t, Y_t), dx),$$

i.e., (W, Y) is a Markov process with transition semigroup $(K_h)_{h\geq 0}$ given by

$$K_h((w, y), \cdot) = \mathcal{N}\left(\begin{pmatrix} w \\ y + hw \end{pmatrix}, \begin{pmatrix} h & h^2/2 \\ h^2/2 & h^3/3 \end{pmatrix}\right), \quad h \geq 0.$$

(c) For the sake of contradiction, suppose that Y is Markov. Since this is a distributional property, we may assume that W is realised on the canonical space for Brownian motion. Then, for any $t, h \geq 0$,

$$E[Y_{t+h}|\mathcal{F}_t^W] = E\left[Y_t + hW_t + \int_t^{t+h}(W_s - W_t)\,ds \,\middle|\, \mathcal{F}_t^W\right] = Y_t + hW_t \quad P\text{-a.s.},$$

where we use Fubini's theorem for the $\mathcal{F}_t^W$-independent integral part. The Markov property then yields that W_t is $\sigma(Y_t)^P$-measurable, where $\sigma(Y_t)^P$ denotes the $(P, \mathcal{F})$-completion of $\sigma(Y_t)$. Consider e.g. $t = 1$. This implies that we can find $\Gamma \in \mathcal{B}(\mathbb{R})$ such that $1_{\{W_1 < 0\}} = 1_{\{Y_1 \in \Gamma\}}$ P-a.s.. From **a)**, we deduce that (W_1, Y_1) is multivariate normal distributed. Thus, because $W_1 \neq Y_1$ we have that $0 < \mathbb{P}(W_1 \geq 0, Y_1 \in \Gamma) = \mathbb{P}(W_1 \geq 0, W_1 < 0)$ which gives a contradiction .

Exercise 5.3

(a) Prove that P-almost all Brownian paths are nowhere on $[0,1]$ Hölder-continuous of order α, for any $\alpha > \frac{1}{2}$.
Hint: Take any $M \in \mathbb{N}$ satisfying $M(\alpha - \frac{1}{2}) > 1$ and show that the set $\{B_{\cdot}(\omega)$ is α-Hölder at some $s \in [0,1]\}$ is contained in the set $\bigcup_{C\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} \bigcup_{k=0,\dots,n-1} \bigcap_{j=1}^{M} \left\{ |B_{\frac{k+j}{n}}(\omega) - B_{\frac{k+j-1}{n}}(\omega)| \leq C\frac{1}{n^\alpha} \right\}$.

(b) The *Kolmogorov-Čentsov theorem* states that a process X on $[0,T]$ satisfying

$$E\big[|X_t - X_s|^\alpha\big] \leq C\,|t-s|^{1+\beta}, \quad s,t \in [0,T],$$

where $\alpha, \beta, C > 0$, has a version which is locally Hölder-continuous of order γ for all $\gamma < \beta/\alpha$. Use this to deduce that Brownian motion has for every $\gamma < 1/2$ a version which is locally Hölder-continuous of order γ.

Remark: One can also show that the Brownian paths are *not* Hölder-continuous of order 1/2. The exact modulus of continuity was found by P. Lévy.

Solution 5.3

(a) Take any $\alpha > \frac{1}{2}$ and let $M \in \mathbb{N}$ satisfying $M(\alpha - \frac{1}{2}) > 1$. If $B_{\cdot}(\omega)$ is Hölder-continuous of order α at the point $s \in [0,1]$, there exists a constant C so that $|B_t(\omega) - B_s(\omega)| \leq C|t-s|^\alpha$ for t near s . Then $|B_{\frac{k}{n}}(\omega) - B_{\frac{k-1}{n}}(\omega)| \leq Cn^{-\alpha}$ for all large enough n, for $\frac{k}{n}$ near s and M successive k's. The set $\{B_{\cdot}(\omega)$ is α-Hölder at some $s \in [0,1]\}$ is therefore contained in

$$\bigcup_{C\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} \bigcup_{k=0,\dots,n-1} \bigcap_{j=1}^{M} \left\{ |B_{\frac{k+j}{n}}(\omega) - B_{\frac{k+j-1}{n}}(\omega)| \leq C\frac{1}{n^\alpha} \right\}; \qquad (*)$$

We show that this is a nullset. As the above Brownian increments are iid $\sim N(0,\frac{1}{n})$, we have, with $Z \sim N(0,1)$, as $P[|Z| \leq \varepsilon] \leq \varepsilon$ for any $\varepsilon \geq 0$, that

$$P\left[\bigcap_{i=1}^{M} \left\{ |B_{\frac{k+j}{n}}(\omega) - B_{\frac{k+j-1}{n}}(\omega)| \leq C\frac{1}{n^\alpha} \right\}\right] = \left(P\left[|Z| \leq \frac{C}{n^{\alpha-1/2}}\right]\right)^M \leq C^M n^{-M(\alpha-\frac{1}{2})}. \tag{2}$$

Now, we have

$$\begin{aligned} D_m := & \bigcap_{n\geq m} \bigcup_{k=0,\dots,n-1} \bigcap_{j=1}^{M} \left\{ |B_{\frac{k+j}{n}}(\omega) - B_{\frac{k+j-1}{n}}(\omega)| \leq C\frac{1}{n^\alpha} \right\} \\ \subseteq & \bigcup_{k=0,\dots,n-1} \bigcap_{j=1}^{M} \left\{ |B_{\frac{k+j}{n}}(\omega) - B_{\frac{k+j-1}{n}}(\omega)| \leq C\frac{1}{n^\alpha} \right\} \quad \text{for each } n \geq m \end{aligned}$$

and therefore, due to (2), as $M(\alpha - \frac{1}{2}) > 1$, we get

$$\begin{aligned} P\big[D_m\big] &\leq \limsup_{n\to\infty} P\left[\bigcup_{k=0,\dots,n-1} \bigcap_{j=1}^{M} \left\{ |B_{\frac{k+j}{n}}(\omega) - B_{\frac{k+j-1}{n}}(\omega)| \leq C\frac{1}{n^\alpha} \right\}\right] \\ &\leq \limsup_{n\to\infty} n\, C^M\, n^{-M(\alpha-\frac{1}{2})} \\ &= 0. \end{aligned}$$

Therefore, being a countable union of nullsets, $P[(*)] = 0$.

(b) Let $Y_\sigma \sim \mathcal{N}(0, \sigma^2)$ for any $\sigma \geq 0$. We note that $E[Y_\sigma^m] = C\sigma^m$, where $C = E[Y_1^m]$. Thus

$$E\left[|B_t - B_s|^{2n}\right] = C_n |t-s|^n \quad \text{for all } n.$$

Writing $\alpha_n := 2n$ and $\beta_n := n-1$ yields that $\frac{\beta_n}{\alpha_n} < \frac{1}{2}$ for any $n \in \mathbb{N}$. Moreover, $\frac{\beta_n}{\alpha_n}$ converges to $1/2$. Thus, we get the result applying the *Kolmogorov-Čentsov theorem*.

Remark: In fact, $E[Y_\sigma^n] = (n-1)!!\,\sigma^n$ for n even and 0 otherwise, where $n!!$ denotes the double factorial, that is the product of every odd number from n to 1.