Coordinator
A. Sepúlveda

Brownian Motion and Stochastic Calculus

Exercise sheet 6

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than April 7th

Exercise 6.1

Let $(B_t)_{t\geq 0}$ be a Brownian motion and define the process $(M_t)_{t\geq 0}$ by $M_t = \sup_{0\leq s\leq t} B_s$. Show that for any fixed $t \geq 0$

$$M_t - B_t \overset{Law}{=} |B_t| \overset{Law}{=} M_t. \tag{1}$$

That is, show that the random variables have the same density functions.

Exercise 6.2 Let $(B_t)_{t\geq 0}$ be a Brownian motion and denote by $\mathcal{G}_t := \sigma(B_u,\, u \leq t)$, $t \geq 0$. Define $\widetilde{R}_0 f(x) = f(x)$ and

$$\widetilde{R}_t f(x) = \frac{1}{\sqrt{2\pi t}} \int_0^\infty f(y) \left[\exp\left(-\frac{1}{2t}(y-x)^2\right) + \exp\left(-\frac{1}{2t}(y+x)^2\right)\right] dy, \quad t > 0.$$

Let us consider the process $(X_t)_{t\geq 0}$ by $X_t := |B_t|$. Show that

$$E\big[f(X_{t+h}) \,\big|\, \mathcal{G}_t\big] = \widetilde{R}_h f(X_t) \;\; P\text{-a.s. for } \; f \in b\mathcal{B}(\mathbb{R}) \text{ and } t, h \geq 0.$$

Exercise 6.3 Let $(B_t)_{t\geq 0}$ be a Brownian motion. For any $a > 0$ consider the stopping times

$$T_a := \inf\big\{t > 0 \,\big|\, B_t \geq a\big\},$$

Show that the Laplace transform of T_a has value:

$$E\big[\exp(-\mu T_a)\big] = \exp\big(-a\sqrt{2\mu}\big), \quad \forall \mu > 0.$$

and show that $P[T_a < \infty] = 1$.
Hint: Consider the martingale $M_t^\lambda = \exp\left(\lambda B_t - \frac{\lambda^2}{2} t\right)$.

Exercise 6.4 Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a decreasing sequence of sub-σ-fields of $\mathcal{F}$ (i.e. $\mathcal{F}_{n+1} \subseteq \mathcal{F}_n \subseteq \mathcal{F}, \forall n \in \mathbb{N}$) and let $(X_n)_{n\in\mathbb{N}}$ be a *backward submartingale*, i.e. $E[|X_n|] < \infty$, X_n is $\mathcal{F}_n$-measurable and $E[X_n \,|\mathcal{F}_{n+1}] \geq X_{n+1}$ P-a.s. for every $n \in \mathbb{N}$.

(a) Show that for any $n \geq m$, $N, M > 0$,

$$E\big[-X_n \mathbf{1}_{\{-X_n \geq M\}}\big] \leq E\big[X_m\big] - E\big[X_n\big] + E\big[|X_m| \mathbf{1}_{\{-X_m \geq N\}}\big] + \frac{N}{M} E\big[X_n^-\big].$$

(b) Show that $\lim\limits_{n\to\infty} E[X_n] > -\infty$ implies that the sequence $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable.
Hint: use a) to conclude that $(X_n^-)_{n\in\mathbb{N}}$ is uniformly integrable.