Brownian Motion and Stochastic Calculus

Exercise sheet 6

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than April 7th

Exercise 6.1

Let $(B_t)_{t\geq 0}$ be a Brownian motion and define the process $(M_t)_{t\geq 0}$ by $M_t = \sup_{0\leq s\leq t} B_s$. Show that for any fixed $t \geq 0$

$$M_t - B_t \overset{Law}{=} |B_t| \overset{Law}{=} M_t. \tag{1}$$

That is, show that the random variables have the same density functions.

Solution 6.1 For any $g : \mathbb{R} \to \mathbb{R}$ bounded Borel measurable function, we know that

$$E\big[g(|B_t|)\big] = \int_{-\infty}^{\infty} g(|x|) \frac{1}{\sqrt{2\pi t}}\, e^{-x^2/(2t)}\, dx = \int_0^{\infty} g(x) \sqrt{\frac{2}{\pi t}}\, e^{-x^2/(2t)}\, dx \tag{2}$$

Thus, we see that the probability density function of $|B_t|$ on $\mathbb{R}$ is given by the function

$$x \mapsto \mathbf{1}_{x\geq 0} \sqrt{\frac{2}{\pi t}}\, e^{-x^2/(2t)}.$$

From Corollary 2.55 of the script, we know that the probability density function of the joint law of (B_t, M_t) where $M_t := \sup_{0\leq s\leq t} B_s$ is given by the function

$$(x,y) \mapsto \frac{2(2y-x)}{\sqrt{2\pi t^3}}\, e^{-(2y-x)^2/(2t)}\, \mathbf{1}_{\{y\geq 0,\, x\leq y\}}. \tag{3}$$

Take any $g : \mathbb{R} \to \mathbb{R}$ bounded Borel measurable function. We deduce from (3) that

$$E\big[g(M_t - B_t)\big] = \int\int_{0\leq y,\, 0\leq y-x} g(y-x)\, \frac{2(2y-x)}{\sqrt{2\pi t^3}}\, e^{-(2y-x)^2/(2t)}\, dx\, dy.$$

By a change of variable $u := y - x$ $v := y$ we get that

$$E\big[g(M_t - B_t)\big] = \int\int_{0\leq u,\, 0\leq v} g(u)\, \sqrt{\frac{2}{\pi t^3}}\, (u+v)\, e^{-(u+v)^2/(2t)}\, du\, dv \tag{4}$$

By another change of variable $n := u$ and $m := u + v$ and as $\int x\, e^{-cx^2/2}\, dx = -\frac{e^{-cx^2/2}}{c}$, we get that

$$\begin{aligned} E\big[g(M_t - B_t)\big] &= \int_0^{\infty} g(n)\, \sqrt{\frac{2}{\pi t^3}} \int_n^{\infty} m\, e^{-m^2/(2t)}\, dm\, dn \\ &= \int_0^{\infty} g(n)\, \sqrt{\frac{2}{\pi t}}\, e^{-n^2/(2t)}\, dn. \end{aligned} \tag{5}$$

Comparing (2) with (5) yields that $M_t - B_t \overset{Law}{=} |B_t|$.
Now, from (3), we deduce for any $g : \mathbb{R} \to \mathbb{R}$ bounded Borel measurable function that

$$E\big[g(M_t)\big] = \int\int_{0\leq y,\, 0\leq y-x} g(y)\, \frac{2(2y-x)}{\sqrt{2\pi t^3}}\, e^{-(2y-x)^2/(2t)}\, dx\, dy.$$

By a change of variable $u := y$ and $v := y - x$ we get that

$$E\big[g(M_t)\big] = \int\int_{0\leq u,\, 0\leq v} g(u)\sqrt{\frac{2}{\pi t^3}}\,(u+v)\,e^{-(u+v)^2/(2t)}\,du\,dv. \tag{6}$$

Comparing (4) with (6) yields $M_t - B_t \overset{Law}{=} M_t$.

Exercise 6.2 Let $(B_t)_{t\geq 0}$ be a Brownian motion and denote by $\mathcal{G}_t := \sigma(B_u,\, u \leq t)$, $t \geq 0$. Define $\widetilde{R}_0 f(x) = f(x)$ and

$$\widetilde{R}_t f(x) = \frac{1}{\sqrt{2\pi t}} \int_0^\infty f(y) \left[\exp\left(-\frac{1}{2t}(y-x)^2\right) + \exp\left(-\frac{1}{2t}(y+x)^2\right)\right] dy, \quad t > 0.$$

Let us consider the process $(X_t)_{t\geq 0}$ by $X_t := |B_t|$. Show that

$$E\big[f(X_{t+h}) \,\big|\, \mathcal{G}_t\big] = \widetilde{R}_h f(X_t) \;\; P\text{-a.s. for } \; f \in b\mathcal{B}(\mathbb{R}) \text{ and } t, h \geq 0.$$

Solution 6.2 Fix any $t, h \geq 0$ and $f \in b\mathcal{B}(\mathbb{R})$. The case where $h = 0$ is trivial, therefore, let $h > 0$. From the lecture (cf. Example 2.23 in Section 3.2 in the lecture notes), we know that Brownian motion is a Markov process with transition semigroup given by $R_0 \tilde{f}(x) = \tilde{f}(x)$ and

$$R_h \tilde{f}(x) = \frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}} \tilde{f}(y) \exp\left(-\frac{(y-x)^2}{2h}\right) dy \quad \text{when} \;\; h > 0, \quad \tilde{f} \in b\mathcal{B}(\mathbb{R}).$$

Therefore, we get for $\tilde{f}(x) := f(|x|) \in b\mathcal{B}(\mathbb{R})$ that

$$\begin{aligned} E\big[f(X_{t+h})\big|\mathcal{G}_t\big] =& R_h \tilde{f}(B_t) \\ =& \frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}} \tilde{f}(y) \exp\left(-\frac{(y-B_t)^2}{2h}\right) dy \\ =& \frac{1}{\sqrt{2\pi h}} \int_{[0,\infty)} f(y) \exp\left(-\frac{(y-B_t)^2}{2h}\right) dy \\ &+ \frac{1}{\sqrt{2\pi h}} \int_{(-\infty,0)} f(-y) \exp\left(-\frac{(y-B_t)^2}{2h}\right) dy \end{aligned} \tag{7}$$

By a change of variables and by observing that $\{0\}$ is a null set, we deduce from (7) that

$$\begin{aligned} E\big[f(X_{t+h})\big|\mathcal{G}_t\big] =& \frac{1}{\sqrt{2\pi h}} \int_{[0,\infty)} f(y) \exp\left(-\frac{(y-B_t)^2}{2h}\right) dy \\ &+ \frac{1}{\sqrt{2\pi h}} \int_{[0,\infty)} f(y) \exp\left(-\frac{(y+B_t)^2}{2h}\right) dy \\ =& \tilde{R}_h f(B_t) \end{aligned} \tag{8}$$

By symmetry of the expression in (8), we see that $E\big[f(X_{t+h})\big|\mathcal{G}_t\big] = \tilde{R}_h f(-B_t)$ and thus

$$E\big[f(X_{t+h})\big|\mathcal{G}_t\big] = \tilde{R}_h f(X_t).$$

Exercise 6.3 Let $(B_t)_{t\geq 0}$ be a Brownian motion. For any $a > 0$ consider the stopping times

$$T_a := \inf\{t > 0 \,|\, B_t \geq a\},$$

Show that the Laplace transform of T_a has value:

$$E\big[\exp(-\mu T_a)\big] = \exp\big(-a\sqrt{2\mu}\big), \quad \forall \mu > 0.$$

and show that $P[T_a < \infty] = 1$.
Hint: Consider the martingale $M_t^\lambda = \exp\Big(\lambda B_t - \frac{\lambda^2}{2}t\Big)$.

Solution 6.3 We know that for any $\lambda \in \mathbb{R}$, the process $M^\lambda := (M_t^\lambda)_{t\geq 0}$ defined by

$$M_t^\lambda = \exp\Big(\lambda B_t - \frac{\lambda^2}{2}t\Big)$$

is a continuous $\mathcal{F}_t$-martingale, where we denote by $(\mathcal{F}_t)_{t\geq 0}$ the filtration generated by B. Moreover, for any $n \in \mathbb{N}$, $T_a \wedge n$ is a bounded stopping time. Thus, applying the stopping theorem (i.e. Serie 6 Exercise 1) we get

$$E\Big[M^\lambda_{T_a\wedge n}\,\Big|\,\mathcal{F}_0\Big] = M_0^\lambda = 1 \quad P\text{-a.s.}$$

By taking expectations, we get that

$$E\Big[M^\lambda_{T_a\wedge n}\Big] = 1$$

Now, on the event $\{T_a < \infty\}$ we have

$$\exp\Big(\lambda B_{T_a\wedge n} - \frac{\lambda^2}{2}(T_a\wedge n)\Big) \overset{n\to\infty}{\longrightarrow} \exp\Big(\lambda B_{T_a} - \frac{\lambda^2}{2}T_a\Big) = e^{\lambda a}\exp\Big(-\frac{\lambda^2}{2}T_a\Big).$$

On the event $\{T_a = \infty\}$ we have $B_t \leq a$ for any $t \geq 0$ and thus we get for any $\lambda > 0$ that

$$\exp\Big(\lambda B_{T_a\wedge n} - \frac{\lambda^2}{2}(T_a\wedge n)\Big)\exp\Big(\lambda B_n - \frac{\lambda^2}{2}n\Big) \overset{n\to\infty}{\longrightarrow} 0.$$

We conclude that for any $\lambda > 0$

$$\exp\Big(\lambda B_{T_a\wedge n} - \frac{\lambda^2}{2}(T_a\wedge n)\Big) \overset{n\to\infty}{\longrightarrow} e^{\lambda a}\exp\Big(-\frac{\lambda^2}{2}T_a\Big)\mathbf{1}_{\{T_a<\infty\}} \quad P\text{-a.s.} \tag{9}$$

Observe that for any $n \in \mathbb{N}$ we have

$$0 \leq \exp\Big(\lambda B_{T_a\wedge n} - \frac{\lambda^2}{2}(T_a\wedge n)\Big) \leq e^{\lambda a}$$

Thus, we deduce from (9), by applying dominated convergence theorem, that for any $\lambda > 0$

$$1 = E\big[M^\lambda_{T_a\wedge n}\big] \overset{n\to\infty}{\longrightarrow} e^{\lambda a}E\Big[\exp\Big(-\frac{\lambda^2}{2}T_a\Big)\mathbf{1}_{\{T_a<\infty\}}\Big]$$

and so, for any $\lambda > 0$

$$e^{\lambda a}E\Big[\exp\Big(-\frac{\lambda^2}{2}T_a\Big)\mathbf{1}_{\{T_a<\infty\}}\Big] = 1. \tag{10}$$

Take any positive, decreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ converging to 0. We deduce from (10) and the monotone convergence theorem that

$$P\big[T_a < \infty\big] = \lim_{n\to\infty} e^{\lambda_n a}E\Big[\exp\Big(-\frac{\lambda_n^2}{2}T_a\Big)\mathbf{1}_{\{T_a<\infty\}}\Big] = 1$$

which proves the first part. Thus, as we now know that $P[T_a < \infty] = 1$ we get from (10) that for any $\lambda > 0$

$$e^{\lambda a} E\left[\exp\left(-\frac{\lambda^2}{2}T_a\right)\right] = 1. \tag{11}$$

Fix any $\mu > 0$. For $\lambda := \sqrt{2\mu}$, (11) yields the desired result.

Exercise 6.4 Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a decreasing sequence of sub-σ-fields of $\mathcal{F}$ (i.e. $\mathcal{F}_{n+1} \subseteq \mathcal{F}_n \subseteq \mathcal{F}, \forall n \in \mathbb{N}$) and let $(X_n)_{n\in\mathbb{N}}$ be a *backward submartingale*, i.e. $E[|X_n|] < \infty$, X_n is $\mathcal{F}_n$-measurable and $E[X_n \,|\mathcal{F}_{n+1}] \geq X_{n+1}$ P-a.s. for every $n \in \mathbb{N}$.

(a) Show that for any $n \geq m$, $N, M > 0$,

$$E\big[-X_n \mathbf{1}_{\{-X_n \geq M\}}\big] \leq E\big[X_m\big] - E\big[X_n\big] + E\big[|X_m|\,\mathbf{1}_{\{-X_m \geq N\}}\big] + \frac{N}{M}E\big[X_n^-\big].$$

(b) Show that $\lim_{n\to\infty} E[X_n] > -\infty$ implies that the sequence $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable.
Hint: use a) to conclude that $(X_n^-)_{n\in\mathbb{N}}$ is uniformly integrable.

Solution 6.4

(a) For any $n \geq m$, $N, M > 0$, by the backward submartingale property, using that the set $\{-X_n > M\} \in \mathcal{F}_n$, we obtain that

$$\begin{aligned}
E\big[-X_n \mathbf{1}_{\{-X_n \geq M\}}\big] &= E\big[-X_n\big] - E[-X_n \mathbf{1}_{\{-X_n < M\}}\big] \\
&\leq E\big[-X_n\big] - E[-X_m \mathbf{1}_{\{-X_n < M\}}\big] \\
&= E\big[-X_n\big] - E\big[-X_m\big] + E[-X_m \mathbf{1}_{\{-X_n \geq M\}}\big] \\
&= E\big[X_m - X_n\big] + E[-X_m \mathbf{1}_{\{-X_n \geq M, -X_m \geq N\}}\big] \\
&\quad + E[-X_m \mathbf{1}_{\{-X_n \geq M, 0 < -X_m < N\}}\big] + E[-X_m \mathbf{1}_{\{-X_n \geq M, -X_m \leq 0\}}\big] \\
&\leq E\big[X_m - X_n\big] + E[-X_m \mathbf{1}_{\{-X_n \geq M, -X_m \geq N\}}\big] + E[-X_m \mathbf{1}_{\{-X_n \geq M, 0 < -X_m < N\}}\big] \\
&\leq E\big[X_m\big] - E\big[X_n\big] + E\big[|X_m|\,\mathbf{1}_{\{-X_m \geq N\}}\big] + \frac{N}{M}E\big[-X_n \mathbf{1}_{\{-X_n \geq M\}}\big] \\
&\leq E\big[X_m\big] - E\big[X_n\big] + E\big[|X_m|\,\mathbf{1}_{\{-X_m \geq N\}}\big] + \frac{N}{M}E\big[-X_n \mathbf{1}_{\{-X_n \geq 0\}}\big] \\
&= E\big[X_m\big] - E\big[X_n\big] + E\big[|X_m|\,\mathbf{1}_{\{-X_m \geq N\}}\big] + \frac{N}{M}E\big[X_n^-\big].
\end{aligned}$$

(b) First, as $(X_n)_{n\in\mathbb{N}}$ is a backward submartingale, we obtain that for any n

$$X_n^+ = X_n + X_n^- \leq E[X_0 \,|\mathcal{F}_n] + X_n^-$$

Thus, as $(E[X_0 \,|\mathcal{F}_n])_{n\in\mathbb{N}}$ is uniformly integrable, if we can show that $(X_n^-)_{n\in\mathbb{N}}$ is uniformly integrable, we can conclude that also $(X_n^+)_{n\in\mathbb{N}}$ is uniformly integrable which then implies that $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable. Now, $(X_n^-)_{n\in\mathbb{N}}$ being uniformly integrable is equivalent having that for any $\varepsilon > 0$ we find $M \geq 0$ such that

$$\sup_{n\in\mathbb{N}} E\big[-X_n \mathbf{1}_{\{-X_n \geq M\}}\big] \leq \varepsilon.$$

Fix any $\varepsilon > 0$. Due to the assumption that $\lim_{n\to\infty} E[X_n] > -\infty$, we can find $m \in \mathbb{N}$ such that for any $n \geq m$

$$E[X_m] - E[X_n] \leq \varepsilon/4.$$

Since X_n is integrable for every $n \in \mathbb{N}$ we can find $M_m \geq 0$ such that for any $M \geq M_m$

$$\max_{n<m} E\big[-X_n \mathbf{1}_{\{-X_n \geq M\}}\big] \leq \varepsilon/4.$$

As

$$\sup_{n\in\mathbb{N}} E\big[-X_n \mathbf{1}_{\{-X_n \geq M\}}\big] \leq \max_{n<m} E\big[-X_n \mathbf{1}_{\{-X_n \geq M\}}\big] + \sup_{n\geq m} E\big[-X_n \mathbf{1}_{\{-X_n \geq M\}}\big]$$

it suffices to find $M \geq M_m$ such that

$$\sup_{n \geq m} E\big[-X_n \, \mathbf{1}_{\{-X_n \geq M\}}\big] \leq 3\varepsilon/4.$$

We first observe that $(X_n^+)_{n\in\mathbb{N}}$ is also a backward submartingale. Indeed, due to Jensen's inequality, we obtain that

$$E\big[X_n^+ \,\big|\, \mathcal{F}_{n+1}\big] \geq E[X_n \,|\, \mathcal{F}_{n+1}]^+ \geq X_{n+1}^+.$$

Now, we claim that $\sup_{n\in\mathbb{N}} E[X_n^-] < \infty$. Assume by contradiction that $\sup_{n\in\mathbb{N}} E[X_n^-] = \infty$. Then, we can find a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} E[X_{n_k}^-] = \sup_{n\in\mathbb{N}} E[X_n^-] = \infty.$$

Using that $(X_n^+)_{n\in\mathbb{N}}$ is also a backward submartingale, we obtain for any k that

$$E\big[X_{n_k}^-\big] = E\big[X_{n_k}^+\big] - E\big[X_{n_k}\big] \leq E\big[X_0^+\big] - E\big[X_{n_k}\big] \leq E\big[|X_0|\big] - E\big[X_{n_k}\big].$$

But as X_0 is integrable and as $\lim_{k\to\infty} E\big[X_{n_k}\big] > -\infty$ by assumption, we obtain that

$$\infty = \lim_{k\to\infty} E[X_{n_k}^-] \leq E\big[|X_0|\big] - \lim_{k\to\infty} E\big[X_{n_k}\big] < \infty$$

which gives us a contradiction. Thus we have proved that $\sup_{u\in\mathbb{N}} E[X_u^-] < \infty$. For any $n \geq m$, $N, M > 0$, by a), by the choice of m and as $\sup_{k\in\mathbb{N}} E[X_k^-] < \infty$, we obtain that

$$\begin{aligned}
E\big[-X_n \, \mathbf{1}_{\{-X_n \geq M\}}\big] &\leq E\big[X_m\big] - E\big[X_n\big] + E\big[|X_m| \, \mathbf{1}_{\{-X_m \geq N\}}\big] + \frac{N}{M} E\big[X_n^-\big] \\
&\leq \varepsilon/4 + E\big[|X_m| \, \mathbf{1}_{\{-X_m \geq N\}}\big] + \frac{N}{M} E\big[X_n^-\big] \\
&\leq \varepsilon/4 + E\big[|X_m| \, \mathbf{1}_{\{-X_m \geq N\}}\big] + \frac{N}{M} \sup_{u\in\mathbb{N}} E\big[X_u^-\big].
\end{aligned}$$

Since X_u is integrable for any $u \in \mathbb{N}$ we can find N big enough such that second term above is smaller than $\varepsilon/4$. After having chosen N we can find $M \geq M_m$ such that that the last term above is smaller than $\varepsilon/4$. Thus, we get for that chosen M that

$$\sup_{n \geq m} E\big[-X_n \, \mathbf{1}_{\{-X_n \geq M\}}\big] \leq 3\varepsilon/4$$

hence we get the result.