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Brownian Motion and Stochastic Calculus

Exercise sheet 7

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than April 14th

Exercise 7.1 Let $(B_t)_{t\in[0,1]}$ be a Brownian motion on $(\Omega, \mathcal{F}, P)$ and define the process $(M_t)_{t\geq 0}$ by $M_t = \sup_{0\leq s\leq t} B_s$. Consider the random variable

$$D = \sup_{0\leq t'\leq 1} \big(\sup_{0\leq t\leq t'} B_t - B_{t'} \big). \tag{1}$$

That is, D characterizes the maximal possible "downfall" in trajectories of the Brownian motion on the time interval $[0, 1]$.

(a) Show that $D \overset{law}{=} \sup_{0\leq t\leq 1} |B_t|$.
Hint: You can use a stronger version of Ex 5-1, which is known as "Lévy's Theorem": The processes $M - B$ and $|B|$ have the saw law under P.

(b) Show that $\sup_{0\leq t\leq 1} |B_t| \overset{law}{=} 1/\sqrt{\bar{T}_1}$, where $\bar{T}_1 = \inf\{t > 0 : |B_t| \geq 1\}$.
Hint: Rewrite $P[\sup_{0\leq t\leq 1} |B_t| \leq x]$ using the self-similarity property of Brownian motion.

(c) Conclude that $E[D] = \sqrt{\pi/2}$.
Hint: For $\sigma > 0$ use the identity

$$\sqrt{2/\pi} \int_0^\infty e^{-x^2/(2\sigma^2)} dx = \sigma,$$

to rewrite the expectation and apply the Laplace transform of $\bar{T}_1$ to conclude the result. Note that you can use the same techniques as Exercise 6.3. to show that

$$\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}\bar{T}_1\right)\right] = \frac{1}{\cosh(\lambda)}$$

Exercise 7.2 For a function $f : [0, \infty) \to \mathbb{R}$, we define its variation $|f| : [0, \infty) \to [0, \infty]$ by

$$|f|(t) := \sup \left\{ \sum_{t_i\in\Pi} \big|f(t_{i+1}) - f(t_i)\big| \,\middle|\, \Pi \text{ is a partition of } [0, t] \right\}.$$

We say that f has finite variation (FV) if $|f|(t) < \infty$ for all $t \geq 0$.

(a) Show that f has finite variation if and only if there exist non decreasing functions $f_1, f_2 : [0, \infty) \to \mathbb{R}$ such that $f = f_1 - f_2$.
Hint: Show that $|f|$ is non decreasing.

Recall that if f is a non decreasing function, then there exists a unique positive measure μ_f on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that $\mu_f\big([0, t]\big) = f(t) - f(0)$ for all $t \geq 0$. Therefore, if f is non decreasing, we call a function $g : [0, \infty) \to \mathbb{R}$ *f-integrable in the Lebesgue–Stieltjes sense* if $\int_0^\infty |g(s)|\, \mu_f(ds) < \infty$. In that case, we define $\int g(s)\, df(s) := \int g(s)\, \mu_f(ds)$ and call it the *Lebesgue–Stieltjes integral.*

(b) Let f be of finite variation and continuous and $g : [0, \infty) \to \mathbb{R}$ such that $\int_0^\infty |g(s)| \, \mu_{|f|}(ds) < \infty$. Show that there are non decreasing, continuous functions $f_1, f_2 : [0, \infty) \to \mathbb{R}$ such that $f = f_1 - f_2$ and both

$$\int_0^\infty |g(s)| \, \mu_{f_1}(ds) < \infty, \quad \int_0^\infty |g(s)| \, \mu_{f_2}(ds) < \infty.$$

Moreover, show that

$$\int g(s) \, df(s) := \int g(s) \, \mu_{f_1}(ds) - \int g(s) \, \mu_{f_2}(ds)$$

is well-defined.

Remark: If f is of finite variation and continuous, we call g *f-integrable in the Lebesgue–Stieltjes sense* if g satisfies $\int_0^\infty |g(s)| \, \mu_{|f|}(ds) < \infty$.