Coordinator
A. Sepúlveda

Brownian Motion and Stochastic Calculus

Exercise sheet 7

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than April 14th

Exercise 7.1 Let $(B_t)_{t\in[0,1]}$ be a Brownian motion on $(\Omega, \mathcal{F}, P)$ and define the process $(M_t)_{t\geq 0}$ by $M_t = \sup_{0\leq s\leq t} B_s$. Consider the random variable

$$D = \sup_{0\leq t'\leq 1} \big(\sup_{0\leq t\leq t'} B_t - B_{t'} \big). \tag{1}$$

That is, D characterizes the maximal possible "downfall" in trajectories of the Brownian motion on the time interval $[0, 1]$.

(a) Show that $D \overset{law}{=} \sup_{0\leq t\leq 1} |B_t|$.
Hint: You can use a stronger version of Ex 5-1, which is known as "Lévy's Theorem": The processes $M - B$ and $|B|$ have the saw law under P.

(b) Show that $\sup_{0\leq t\leq 1} |B_t| \overset{law}{=} 1/\sqrt{\bar{T}_1}$, where $\bar{T}_1 = \inf\{t > 0 : |B_t| \geq 1\}$.
Hint: Rewrite $P[\sup_{0\leq t\leq 1} |B_t| \leq x]$ using the self-similarity property of Brownian motion (cf. Proposition 1.1 (3) in Section 2.1 of the lecture notes).

(c) Conclude that $E[D] = \sqrt{\pi/2}$.
Hint: For $\sigma > 0$ use the identity

$$\sqrt{2/\pi} \int_0^\infty e^{-x^2/(2\sigma^2)} dx = \sigma,$$

to rewrite the expectation and apply the Laplace transform of $\bar{T}_1$ (cf. Ex 4-2) to conclude the result.

Solution 7.1

(a) Let $Z_t := M_t - B_t$ and $Y_t := |B_t|$. With the definition of D we have to check that

$$\sup_{0\leq t\leq 1} Z_t \overset{law}{=} \sup_{0\leq t\leq 1} Y_t.$$

Since both Z and Y are continuous processes, it suffices to check that

$$\sup_{t\in[0,1]\cap\mathbb{Q}} Z_t \overset{law}{=} \sup_{t\in[0,1]\cap\mathbb{Q}} Y_t. \tag{2}$$

Let $(t_n)_{n\in\mathbb{N}}$ be a counting sequence in $[0, 1] \cap \mathbb{Q}$. By Lévy's Theorem, the processes Z and Y have the same law, and therefore for $n \in \mathbb{N}$ the random variables

$$Z_n := \sup(Z_{t_1}, Z_{t_2}, \ldots, Z_{t_n}) \quad Y_n := \sup(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n})$$

have the same law. Since Z_n and Y_n converge monotonically to $\sup_{t\in[0,1]\cap\mathbb{Q}} Z_t$ and $\sup_{t\in[0,1]\cap\mathbb{Q}} Y_t$ we have for all $x \in \mathbb{R}$

$$\begin{aligned} P\left[\sup_{t\in[0,1]\cap\mathbb{Q}} Z_t \le x\right] =& P\left[\bigcap_{n=0}^{\infty}\{Z_n \le x\}\right] \\ =& \lim_{n\to\infty} P[Z_n \le x] \\ =& \lim_{n\to\infty} P[Y_n \le x] \\ =& P\left[\sup_{t\in[0,1]\cap\mathbb{Q}} Y_t \le x\right], \end{aligned}$$

which yields (2).

(b) We recall the self-similarity property of Brownian motion, i.e., for $c > 0$

$$cB_{t/c^2} \overset{law}{=} B_t.$$

Therefore, for $x > 0$

$$\begin{aligned} P[\sup_{0\le t\le 1} |B_t| \le x] =& P[\sup_{0\le t\le 1} |B_{t/x^2}| \le 1] \\ =& P[\sup_{0\le t\le 1/x^2} |B_t| \le 1] \\ =& P[\bar{T}_1 \ge x^{-2}] \\ =& P[1/\sqrt{\bar{T}_1} \le x]. \end{aligned}$$

(c) Using the identity

$$\sqrt{2/\pi}\int_0^\infty e^{-x^2/(2\sigma^2)}dx = \sigma$$

and Tonelli's Theorem we have

$$\begin{aligned} E[D] =& E[\sup_{0\le t\le 1} |B_t|] \\ =& E[1/\sqrt{\bar{T}_1}] \\ =& \sqrt{2/\pi}\int_0^\infty E[e^{-x^2\bar{T}_1/2}]dx. \end{aligned}$$

From Ex 4-2 we know that the Laplace transform of $\bar{T}_1$ is

$$E[e^{-\mu\bar{T}_1}] = 1/cosh(\sqrt{2\mu}), \quad \forall \mu > 0.$$

Putting everything together, we have

$$\begin{aligned} E[D] =& \sqrt{2/\pi}\int_0^\infty \frac{dx}{cosh(x)} \\ =& 2\sqrt{2/\pi}\int_0^\infty \frac{e^x dx}{e^{2x}+1} \\ =& 2\sqrt{2/\pi}\int_1^\infty \frac{dy}{y^2+1} \\ =& 2\sqrt{\frac{2}{\pi}}\cdot\frac{\pi}{4} = \sqrt{\frac{\pi}{2}}. \end{aligned}$$

Exercise 7.2 For a function $f : [0, \infty) \to \mathbb{R}$, we define its variation $|f| : [0, \infty) \to [0, \infty]$ by

$$|f|(t) := \sup \left\{ \sum_{t_i \in \Pi} \big| f(t_{i+1}) - f(t_i) \big| \, \middle| \, \Pi \text{ is a partition of } [0, t] \right\}.$$

We say that f has finite variation (FV) if $|f|(t) < \infty$ for all $t \geq 0$.

(a) Show that f has finite variation if and only if there exist non decreasing functions $f_1, f_2 : [0, \infty) \to \mathbb{R}$ such that $f = f_1 - f_2$.
Hint: Show that $|f|$ is non decreasing.

Recall that if f is a non decreasing and continuous function, then there exists a unique positive measure μ_f on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that $\mu_f\big([0, t]\big) = f(t) - f(0)$ for all $t \geq 0$. Therefore, if f is non decreasing and continuous, we call a function $g : [0, \infty) \to \mathbb{R}$ *f-integrable in the Lebesgue–Stieltjes sense* if $\int_0^\infty |g(s)| \, \mu_f(ds) < \infty$. In that case, we define $\int g(s) \, df(s) := \int g(s) \, \mu_f(ds)$ and call it the *Lebesgue–Stieltjes integral*.

(b) Let f be of finite variation and continuous and $g : [0, \infty) \to \mathbb{R}$ such that $\int_0^\infty |g(s)| \, \mu_{|f|}(ds) < \infty$. Show that there are non decreasing, continuous functions $f_1, f_2 : [0, \infty) \to \mathbb{R}$ such that $f = f_1 - f_2$ and both

$$\int_0^\infty |g(s)| \, \mu_{f_1}(ds) < \infty, \quad \int_0^\infty |g(s)| \, \mu_{f_2}(ds) < \infty.$$

Moreover, show that

$$\int g(s) \, df(s) := \int g(s) \, \mu_{f_1}(ds) - \int g(s) \, \mu_{f_2}(ds)$$

is well-defined.

Hint: Recall that if f has finite variation and continuous, then $|f|$ is continuous.

Remark: If f is of finite variation and continuous, we call g *f-integrable in the Lebesgue–Stieltjes sense* if g satisfies $\int_0^\infty |g(s)| \, \mu_{|f|}(ds) < \infty$.

Solution 7.2

(a) For the one direction, let $f : [0, \infty) \to \mathbb{R}$ be of finite variation. We define

$$f_1(t) := \frac{|f|(t) + f(t)}{2} \quad \text{and} \quad f_2(t) := \frac{|f|(t) - f(t)}{2},$$

where $|f|(t)$ denotes the variation of f. We claim that both f_1, f_2 are non decreasing. We first show that $|f|$ is non decreasing. For that purpose, fix any $0 \leq s < t$ and denote by Π_s and Π_t the set of finite partition of $[0, s]$ and $[0, t]$, respectively. Moreover, let $\Pi_{s;t}$ be the set of finite partitions of $[0, t]$ such that s is a grid point, i.e. $s = t_i$ for some t_i. Then, we see that

$$|f|(s) \leq \sup_{\Pi_{s;t}} \sum_{t_i \in \Pi} \big| f(t_{i+1}) - f(t_i) \big| \leq \sup_{\Pi_t} \sum_{t_i \in \Pi} \big| f(t_{i+1}) - f(t_i) \big| = |f|(t).$$

So we conclude that $|f|$ is indeed non decreasing. We denote by $\Pi_{[s,t]}$ the set of finite partitions of $[s, t]$. Then, we have

$$\sup_{\Pi_{s;t}} \sum_{t_i \in \Pi} \big| f(t_{i+1}) - f(t_i) \big| = \sup_{\Pi_s} \sum_{t_i \in \Pi} \big| f(t_{i+1}) - f(t_i) \big| + \sup_{\Pi_{[s,t]}} \sum_{t_i \in \Pi} \big| f(t_{i+1}) - f(t_i) \big|.$$

Now, to see that f_1 is non decreasing, it suffices to show that for any $0 \leq s < t$,

$$|f|(t) - |f|(s) \geq -\big(f(t) - f(s)\big).$$

To see this, observe that

$$\begin{aligned} |f|(t) - |f|(s) &\geq \sup_{\Pi_{s;t}} \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| - \sup_{\Pi_s} \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| \\ &= \sup_{\Pi_{[s,t]}} \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| \geq -\big(f(t) - f(s)\big). \end{aligned}$$

To show that f_2 is non decreasing, it suffices to show that

$$|f|(t) - |f|(s) \geq f(t) - f(s),$$

which we get by applying the same argument as for f_1.
For the other direction, let $f = f_1 - f_2$, where f_1 and f_2 are non decreasing. Then we have for any $t \geq 0$ and any partition Π of $[0,t]$ that

$$\begin{aligned} \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| &= \sum_{t_i \in \Pi} \Big|\big(f_1(t_{i+1}) - f_2(t_{i+1})\big) - \big(f_1(t_i) - f_2(t_i)\big)\Big| \\ &\leq \sum_{t_i \in \Pi} \big|f_1(t_{i+1}) - f_1(t_i)\big| + \sum_{t_i \in \Pi} \big|f_2(t_{i+1}) - f_2(t_i)\big| \\ &\leq f_1(t) - f_1(0) + f_2(t) - f_2(0). \end{aligned}$$

Thus, we see that for every $t \geq 0$

$$|f|(t) = \sup_{\Pi} \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| < \infty.$$

Remark: f_1, f_2 are not unique.

(b) Define $f_1 = |f|$ and $f_2 := |f| - f$. We deduce from **a)** that they are both non decreasing and of course $f = f_1 - f_2$. Moreover, due to the hint, both f_1 and f_2 are continuous. Let μ_{f_1} and μ_{f_2} be the corresponding measures defined in the exercise. Since $\mu_{f_2}\big([0,t]\big) = f_2(t) \leq 2|f|(t) = 2\mu_{|f|}([0,t])$, for every $t \geq 0$, we conclude that $\mu_{f_2} \leq 2\mu_{|f|}$ on $\mathcal{B}\big([0,\infty)\big)$. Thus, we see that

$$\int_0^\infty |g(s)|\, \mu_{f_1}(ds) < \infty, \quad \int_0^\infty |g(s)|\, \mu_{f_2}(ds) < \infty.$$

Therefore,

$$\int_0^\infty g(s)\, \mu_{f_1}(ds) - \int_0^\infty g(s)\, \mu_{f_2}(ds)$$

is well-defined. To show that $\int g\, df$ is well-defined, it remains to show that the definition

$$\int g(s)\, df(s) := \int g(s)\, \mu_{f_1}(ds) - \int g(s)\, \mu_{f_2}(ds)$$

is independent of the choice of f_1, f_2. Let $\overline{f}_1, \overline{f}_2$ be two other functions with the desired properties. Fix any $t \geq 0$. We have for $g(s) := \mathbf{1}_{[0,t]}(s)$ that for any $u \geq 0$

$$\begin{aligned} \int_0^u g(s)\, \mu_{\overline{f}_1}(ds) - \int_0^u g(s)\, \mu_{\overline{f}_2}(ds) &= \overline{f}_1(t \wedge u) - \overline{f}_1(0) - \overline{f}_2(t \wedge u) + \overline{f}_2(0) \\ &= f(t \wedge u) - f(0) \\ &= f_1(t \wedge u) - f_2(t \wedge u) + f_2(0) \\ &= \int_0^u g(s)\, \mu_{f_1}(ds) - \int_0^u g(s)\, \mu_{f_2}(ds). \end{aligned}$$

Thus, we conclude that the same holds true for $g(s) := \mathbf{1}_A$, where $A \in \mathcal{B}(\mathbb{R}_+)$. We conclude that the same holds true for general g Borel by measure theoretical induction.