Coordinator
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Brownian Motion and Stochastic Calculus

Exercise sheet 8

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than April 28th

Exercise 8.1 Recall that for $M, N \in \mathcal{M}^c_{0,\mathrm{loc}}$, the *quadratic covariation process* $\langle M, N \rangle$ is defined by

$$\langle M, N \rangle = \frac{1}{4} \left(\langle M + N \rangle - \langle M - N \rangle \right).$$

1. Show that $\langle M, N \rangle$ is the unique (up to indistiguishability) continuous process B of finite variation with $B_0 = 0$ such that $MN - B \in \mathcal{M}^c_{0,\mathrm{loc}}$.
   Hint: Use Proposition 4.(1.4) in the lecture notes.
   Remark: As an immediate consequence of a), $\langle \cdot, \cdot \rangle$ is bilinear.

2. Show that for any stopping time τ,

   $$\langle M^\tau, N \rangle = \langle M, N^\tau \rangle = \langle M, N \rangle^\tau$$

   (again, up to indistinguishability).

Exercise 8.2 Let $(\Omega, \mathcal{G}, \mathcal{G}_t, P)$ satisfying the usual conditions.

(a) Show that every continuous *bounded* local martingale is a martingale.

(b) Let $0 < T < \infty$ be a deterministic time. Show that any nonnegative continuous local martingale $(X_t)_{t \in [0,T]}$ with $E[X_0] < \infty$ is also a supermartingale, and if

$$E[X_T] = E[X_0],$$

then $(X_t)_{t \in [0,T]}$ is a martingale.

Exercise 8.3

(a) For any $M \in \mathcal{M}^c_{0,\mathrm{loc}}$, define as usual $M^*_t := \sup_{0 \le s \le t} |M_s|$ for $t \ge 0$. Prove that for any $t \ge 0$ and $C, K > 0$, we have

$$P\big[M^*_t > C\big] \le \frac{4K}{C^2} + P\big[\langle M \rangle_t > K\big].$$

Hint: First stop M to make it bounded; then stop $\langle M \rangle$ and use the Tchebycheff and Doob inequalities (remember that the constant in Doob's inequality for fixed $p > 1$, denoted by C_p, is equal to $\left(\frac{p}{p-1}\right)^p$).
Remark: Intuitively, this means that one can control the running supremum of M in terms of the quadratic variation of M.

(b) Let M be a right-continuous local martingale null at 0. Show that there is a localizing sequence $(\tau_n)_{n \in \mathbb{N}}$ such that M^{τ_n} is a uniformly integrable martingale for each n.