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Brownian Motion and Stochastic Calculus

Exercise sheet 8

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than April 28th

Exercise 8.1 Recall that for $M, N \in \mathcal{M}^c_{0,\text{loc}}$, the *quadratic covariation process* $\langle M, N\rangle$ is defined by

$$\langle M, N\rangle = \frac{1}{4}\left(\langle M+N\rangle - \langle M-N\rangle\right).$$

1. Show that $\langle M, N\rangle$ is the unique (up to indistiguishability) continuous process B of finite variation with $B_0 = 0$ such that $MN - B \in \mathcal{M}^c_{0,\text{loc}}$.
   Hint: Use Proposition 4.(1.4) in the lecture notes.
   Remark: As an immediate consequence of a), $\langle\cdot,\cdot\rangle$ is bilinear.

2. Show that for any stopping time τ,

$$\langle M^\tau, N\rangle = \langle M, N^\tau\rangle = \langle M, N\rangle^\tau$$

   (again, up to indistinguishability).

Solution 8.1

(a) It is clear that $\langle M, N\rangle_0 = 0$ and that $\langle M, N\rangle$ is of finite variation, given that is a sum of monotone processes. Note that the sum of local martingale is a local martingale. Then $MN - \frac{1}{4}\left(\langle M+N\rangle - \langle M-N\rangle\right)$ is a local martingale because $\frac{1}{4}MN = (M+N)^2 - (M-N)^2$.

To see that it is unique note that if B, B' are continuous processes of finite variation with $B_0, B'_0 = 0$ such that $MN - B$ and $MN - B'$ are martingales, note that $(MN - B) - (MN - B') = B - B'$ is a martingale with finite variation. Proposition (1.4) implies that $B = B'$.

(b) From (a) it is enough to show that $M^\tau N - \langle M, N\rangle^\tau = (M^\tau N^\tau - \langle M, N\rangle^\tau) + M^\tau(N - N^\tau)$ is a local martingale. The first term is a local martingale by definition so it is just enough to show that $M^\tau(N - N^\tau)$ is a local martingale. For this take $\tau_n \nearrow \infty$ such that for all $n \in \mathbb{N}$, M^{τ_n} and N^{τ_n} are bounded martingales. Now, take $s \leq t$, note that thanks to Exercise 4.2 and the fact that $M^{\tau_n}_{t\wedge\tau}(N^{\tau_n}_t - N^{\tau_n}_{t\wedge\tau})$ is 0 when $\tau \geq t$,

$$\begin{aligned}
\mathbb{E}\left[M^{\tau_n}_{t\wedge\tau}(N^{\tau_n}_t - N^{\tau_n}_{t\wedge\tau}) \mid \mathcal{F}_s\right] &= \mathbb{E}\left[\mathbb{E}\left[M^{\tau_n}_\tau(N^{\tau_n}_t - N^{\tau_n}_\tau)\mathbf{1}_{\tau\leq t} \mid \mathcal{F}_{\tau\vee s}\right] \mid \mathcal{F}_s\right] \\
&= \mathbb{E}\left[M^{\tau_n}_\tau(N^{\tau_n}_{s\vee\tau} - N^{\tau_n}_\tau)\mathbf{1}_{\tau\leq t} \mid \mathcal{F}_s\right] \\
&= \mathbb{E}\left[M^{\tau_n}_\tau(N^{\tau_n}_s - N^{\tau_n}_\tau)\mathbf{1}_{\tau\leq s} \mid \mathcal{F}_s\right] \\
&= M^{\tau_n}_\tau(N^{\tau_n}_s - N^{\tau_n}_\tau)\mathbf{1}_{\tau\leq s} = M^{\tau_n}_{s\wedge\tau}(N^{\tau_n}_s - N^{\tau_n}_{s\wedge\tau}),
\end{aligned}$$

because, as seen in Exercise 4.2 if $X \in \mathcal{F}_\tau$, $X\mathbf{1}_{\tau\leq s} \in \mathcal{F}_s$. Thus, $M^{\tau_n}_{t\wedge\tau}(N^{\tau_n}_t - N^{\tau_n}_{t\wedge\tau})$ is a martingale, which let us conclude.

Exercise 8.2 Let $(\Omega, \mathcal{G}, \mathcal{G}_t, P)$ satisfying the usual conditions.

(a) Show that every continuous *bounded* local martingale is a martingale.

(b) Let $0 < T < \infty$ be a deterministic time. Show that any nonnegative continuous local martingale $(X_t)_{t\in[0,T]}$ with $E[X_0] < \infty$ is also a supermartingale, and if

$$E[X_T] = E[X_0],$$

then $(X_t)_{t\in[0,T]}$ is a martingale.

Solution 8.2

(a) Without loss of generality, suppose $(X_t)_{t\geq 0}$ is a local martingale with $X_0 = 0$ and let B be a constant such that $|X_t| \leq B$ for all $t \geq 0$. Let $(\tau_k)_{k\in\mathbb{N}}$ be a localizing sequence for X, i.e. it is a non decreasing sequence of stopping times such that $(X_{t\wedge\tau_k})_{t\geq 0}$ is a martingale for any k and $\tau_k \nearrow +\infty$ a.s. Fix $s \leq t$, by the martingale property we have

$$E\big[X_{t\wedge\tau_k} \,\big|\, \mathcal{G}_s\big] = X_{s\wedge\tau_k} \quad \text{a.s.}$$

By dominated convergence theorem, which we can apply by the uniform boundedness of X, we get that

$$E\big[X_t \,\big|\, \mathcal{G}_s\big] = \lim_{k\to\infty} E\big[X_{t\wedge\tau_k} \,\big|\, \mathcal{G}_s\big] = \lim_{k\to\infty} X_{s\wedge\tau_k} = X_s \quad \text{a.s.}$$

(b) Let $(\tau_k)_{k\in\mathbb{N}}$ be a localizing sequence for X (see **a)** for the definition). Then, applying the local martingale property, we have for any $0 \leq s \leq t \leq T$ that

$$X_{s\wedge\tau_k} = E[X_{t\wedge\tau_k}|\mathcal{G}_s] \quad \text{a.s.}$$

Since X is nonnegative, we can apply Fatou's lemma to get for any $0 \leq s \leq t \leq T$ that

$$X_s = \lim_{k\to\infty} X_{s\wedge\tau_k} = \liminf_{k\to\infty} E\big[X_{t\wedge\tau_k} \,\big|\, \mathcal{G}_s\big] \geq E\big[\liminf_{k\to\infty} X_{t\wedge\tau_k} \,\big|\, \mathcal{G}_s\big] = E\big[X_t|\mathcal{G}_s\big] \quad \text{a.s.} \tag{1}$$

Moreover as X is nonnegative, we obtain by applying Fatou's Lemma that for any $t \in [0, T]$

$$E\big[X_t\big] = E\big[\liminf_{k\to\infty} X_{t\wedge\tau_k}\big] \leq \liminf_{k\to\infty} E\big[X_{t\wedge\tau_k}\big] = E\big[X_0\big] < \infty$$

and so $(X_t)_{t\geq 0}$ is a supermartingale.
Now, take the expectation on both sides in (1), we get

$$E[X_s] \geq E[X_t]$$

for all $0 \leq s \leq t \leq T$. In particular, using monotonicity of the expectation for a supermartingale, we have

$$E[X_0] \geq E[X_s] \geq E[X_t] \geq E[X_T] \quad \text{for all } 0 \leq s \leq t \leq T. \tag{2}$$

Using the assumption $E[X_T] = E[X_0]$, we see that the previous inequalities in (2) are all equalities. If the inequality in (1) was strict on a set of positive probability, we would have $E[X_s] > E[X_t]$, which gives a contradiction, and so the equality must hold with probability one. Thus, X is a martingale.

Exercise 8.3

(a) For any $M \in \mathcal{M}^c_{0,\text{loc}}$, define as usual $M^*_t := \sup_{0 \le s \le t} |M_s|$ for $t \ge 0$. Prove that for any $t \ge 0$ and $C, K > 0$, we have

$$P\big[M^*_t > C\big] \le \frac{4K}{C^2} + P\big[\langle M \rangle_t > K\big].$$

Hint: First stop M to make it bounded; then stop $\langle M \rangle$ and use the Tchebycheff and Doob inequalities (remember that the constant in Doob's inequality for fixed $p > 1$, denoted by C_p, is equal to $\big(\frac{p}{p-1}\big)^p$).
Remark: Intuitively, this means that one can control the running supremum of M in terms of the quadratic variation of M.

(b) Let M be a right-continuous local martingale null at 0. Show that there is a localizing sequence $(\tau_n)_{n \in \mathbb{N}}$ such that M^{τ_n} is a uniformly integrable martingale for each n.

Solution 8.3

(a) For $K > 0$, we consider the stopping time $\sigma_K := \inf\{t > 0 | \langle M \rangle_t > K\}$. Since $\langle M \rangle$ is continuous, we have that $\langle M \rangle_t \le K$ for $t \le \sigma_K$, and therefore

$$E\big[\langle M^{\sigma_K} \rangle_\infty\big] = E\big[\langle M \rangle_{\sigma_K}\big] \le K.$$

Hence, Exercise **8-1 a)** gives that $M^{\sigma_K} \in \mathcal{H}^{2,c}_0$. We can therefore apply Tchebycheff's and Doob's inequality (and use that the constant in Doob's inequality for fixed $p > 1$, denoted by C_p, is equal to $\big(\frac{p}{p-1}\big)^p$), obtaining that

$$\begin{aligned} P\big[(M^{\sigma_K})^*_t > C\big] &\le \frac{E\big[((M^{\sigma_K})^*_t)^2\big]}{C^2} \\ &\le \frac{4E\big[(M^{\sigma_K})^2_t\big]}{C^2} \\ &= \frac{4E\big[\langle M^{\sigma_K} \rangle_t\big]}{C^2} \\ &\le \frac{4K}{C^2}. \end{aligned}$$

To obtain the claim, we observe that

$$\{M^{\sigma_K}_t \ne M_t\} \subseteq \{\sigma_K < t\} = \{\langle M \rangle_t > K\},$$

which finally implies that

$$\begin{aligned} P\big[M^*_t > C\big] &= P\big[M^*_t > C, \sigma_K \ge t\big] + P\big[M^*_t > C, \sigma_K < t\big] \\ &\le \frac{4K}{C^2} + P\big[\langle M \rangle_t > K\big]. \end{aligned}$$

(b) Since $M \in \mathcal{M}_{0,loc}$, there is a localizing sequence $(\sigma_n)_{n \in \mathbb{N}}$ such that M^{σ_n} is a martingale for each n. Consider the sequence of stopping times $\tau_n := \sigma_n \wedge n,\ n \ge 0$. By construction, $\tau_n \uparrow \infty$ P-a.s. and $M^{\tau_n} = (M^{\sigma_n})^n$ is a martingale for each n due to Exercise 4.2. Moreover, from the stopping theorem, as $n \wedge \sigma_n$ is bounded, we have that $M_{\tau_n \wedge t} = E[M_{n \wedge \sigma_n} | \mathcal{F}_t]$ for every $t \ge 0$. As $M_{n \wedge \sigma_n} = E[M_n \,|\, \mathcal{F}_{n \wedge \sigma_n}]$ a.s., we see that $M_{n \wedge \sigma_n} \in \mathcal{L}^1$. Thus, we deduce the uniform integrability of M^{τ_n} because for all $t \ge 0$, $|M^{\tau_n}_t| \le |M^*_{\tau_n}|$ and, thanks to the last part $|M^*_{\tau_n}|$ is integrable.