Brownian Motion and Stochastic Calculus

Exercise sheet 9

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than May 5th

Exercise 9.1 For $M \in \mathcal{M}_{0,loc}^c$, we denote by $L_{loc}^2(M)$ the space of all predictable processes for which there is a sequence of stopping times $\tau_n \uparrow \infty$ P-a.s. such that $E\left[\int_0^{\tau_n} H_s^2 \, d\langle M\rangle_s\right] < \infty$ for any n.

(a) Let H be predictable. Show that

$$H \in L_{loc}^2(M) \iff \int_0^t H_s^2 \, d\langle M\rangle_s < \infty \quad P\text{-a.s. for each } t \geq 0.$$

(b) Show that for any continuous semimartingale X, any adapted RCLL process H and any sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$ of $[0,\infty)$ with $\lim_{n\to\infty} |\Pi_n| = 0$, we have

$$\int_0^t H_{s-} \, dX_s = \lim_{n\to\infty} \sum_{t_i \leq t, t_i \in \Pi_n} H_{t_i} \left(X_{t_{i+1}\wedge t} - X_{t_i}\right) \quad \text{in probability.}$$

(c) Find an adapted process with RCLL paths which is not locally bounded.

Exercise 9.2 Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions and let $\sigma \leq \tau$ be two stopping times. Moreover, let Z be a bounded, $\mathcal{F}_\sigma$-measurable random variable. The goal of this exercise is to compute the stochastic integral process $\int Z 1_{]\!]\sigma,\tau]\!]} \, dM$ for an integrator $M \in \mathcal{M}_{0,\mathrm{loc}}^c$.

(a) For a (uniformly integrable) right-continuous martingale $X = (X_t)_{t\geq 0}$, show that the process $Z(X^\tau - X^\sigma)$ is again a (uniformly integrable) right-continuous martingale.
Hint: Use Lemma 4.1.19 from the lecture notes to show the assertion first for Z of the form $Z = 1_A$ for some $A \in \mathcal{F}_\sigma$. Then extend the result to general Z.

(b) Let $M, N \in \mathcal{M}_{0,\mathrm{loc}}^c$. Show that

$$\langle Z(M^\tau - M^\sigma), N\rangle = Z\langle M^\tau - M^\sigma, N\rangle = Z(\langle M, N\rangle^\tau - \langle M, N\rangle^\sigma).$$

(c) Let $M \in \mathcal{M}_{0,\mathrm{loc}}^c$ and set $H := Z 1_{]\!]\sigma,\tau]\!]}$. Show that $H \in L_{\mathrm{loc}}^2(M)$ and

$$\int H \, dM = Z(M^\tau - M^\sigma).$$

Conclude that if M is a (uniformly integrable) martingale, then the stochastic integral $\int H \, dM$ is also a (uniformly integrable) martingale.
Remark: The last statement is **not** true for arbitrary bounded $H \in L_{\mathrm{loc}}^2(M)$.

Exercise 9.3 Let $(B_t)_{t\geq 0}$ be a Brownian motion. Fix any $0 < T < \infty$ and let $f \in L^2([0,T])$ be a deterministic function. For any $0 \leq a < b \leq T$ we set

$$\mathcal{J}_{a,b} := \int_a^b f(s) \, dB_s.$$

Moreover, for any $t \in [0,T]$, we denote $\mathcal{J}_t := \mathcal{J}_{0,t}$.

(a) Show that the process $\mathcal{J} := (\mathcal{J}_{a,b})_{0 \leq a \leq b \leq T}$ is a centered Gaussian process and calculate its covariance function.

(b) Show that the process $(\mathcal{J}_t)_{t \in [0,T]}$ has the same law as the process $Y := (Y_t)_{t \in [0,T]}$ defined by

$$Y_t := B_{\int_0^t f^2(s)\, ds}.$$