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Brownian Motion and Stochastic Calculus

Exercise sheet 9

Please hand in your solutions during exercise class or in your assistant's box in HG E65 no latter than May 5th

Exercise 9.1 For $M \in \mathcal{M}^c_{0,loc}$, we denote by $L^2_{loc}(M)$ the space of all predictable processes for which there is a sequence of stopping times $\tau_n \uparrow \infty$ P-a.s. such that $E\Big[\int_0^{\tau_n} H_s^2 \, d\langle M\rangle_s\Big] < \infty$ for any n.

(a) Let H be predictable. Show that

$$H \in L^2_{loc}(M) \iff \int_0^t H_s^2 \, d\langle M\rangle_s < \infty \quad P\text{-a.s. for each } t \geq 0.$$

(b) Show that for any continuous semimartingale X, any adapted RCLL process H and any sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$ of $[0,\infty)$ with $\lim_{n\to\infty} |\Pi_n| = 0$, we have

$$\int_0^t H_{s-} \, dX_s = \lim_{n\to\infty} \sum_{t_i \leq t, t_i \in \Pi_n} H_{t_i}\left(X_{t_{i+1}\wedge t} - X_{t_i}\right) \quad \text{in probability.}$$

(c) Find an adapted process with RCLL paths which is not locally bounded.

Solution 9.1

(a) ($\Rightarrow$) Let $H \in L^2_{loc}(M)$ and let $(\tau_n)_{n\in\mathbb{N}}$ be the corresponding localizing sequence. By definition, we have for each n that

$$P\Big[\int_0^{\tau_n} H_s^2 \, d\langle M\rangle_s < \infty\Big] = 1$$

Fix any $t \geq 0$. We see that for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
&P\Big[\int_0^t H_s^2 \, d\langle M\rangle_s = \infty\Big] \\
&= P\Big[\Big\{\int_0^t H_s^2 \, d\langle M\rangle_s = \infty\Big\} \cap \Big\{\tau_n \leq t\Big\}\Big] + P\Big[\Big\{\int_0^t H_s^2 \, d\langle M\rangle_s = \infty\Big\} \cap \Big\{\tau_n > t\Big\}\Big] \\
&\leq P\big[\tau_n \leq t\big] + P\Big[\Big\{\int_0^{\tau_n} H_s^2 \, d\langle M\rangle_s = \infty\Big\} \cap \Big\{\tau_n > t\Big\}\Big] \\
&\leq P\big[\tau_n \leq t\big] + P\Big[\int_0^{\tau_n} H_s^2 \, d\langle M\rangle_s = \infty\Big] \\
&= P\big[\tau_n \leq t\big].
\end{aligned}$$

Thus, we conclude that

$$P\Big[\int_0^t H_s^2 \, d\langle M\rangle_s = \infty\Big] \leq \lim_{n\to\infty} P\big[\tau_n \leq t\big] = 0.$$

($\Leftarrow$) Let H be predictable such that

$$\int_0^t H_s^2 \, d\langle M\rangle_s < \infty \quad P\text{-a.s. for each } t \geq 0. \tag{1}$$

Consider the sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ defined by

$$\tau_n := \inf\Big\{t \geq 0 \,\Big|\, \int_0^t H_s^2 \, d\langle M\rangle_s > n\Big\}.$$

Due to (1), we obtain that $\tau_n \uparrow \infty$ P-a.s. Moreover, due to the definition of τ_n and the (left)-continuity of $\int H \, d\langle M\rangle$, we have for each n

$$E\bigg[\int_0^{\tau_n} H_s^2 \, d\langle M\rangle_s\bigg] \leq n < \infty.$$

(b) For each n, set

$$H^n := \sum_{t_i \in \Pi_n} H_{t_i} \, \mathbf{1}_{(t_i, t_{i+1}]}.$$

By definition, we obtain that for each $t \geq 0$,

$$(H^n \cdot X)_t := \int_0^t H_s^n \, dX_s = \sum_{t_i \leq t, t_i \in \Pi_n} H_{t_i} \, (X_{t_{i+1}\wedge t} - X_{t_i}).$$

Set $\mathcal{H}^n := H^n - H_-$. Being left-continuous and adapted, each H^n and H_- are locally bounded and thus each $\mathcal{H}^n$ is locally bounded, too. Moreover, $\mathcal{H}^n \to 0$ pointwise on $\Omega \times [0, \infty)$ as $n \to \infty$. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $|\mathcal{H}^n| \leq \varepsilon$ for all $n \geq N$. Then, we have

$$|\mathcal{H}^n| \leq \varepsilon + \max_{n=1,\dots,N} |\mathcal{H}^n|,$$

which is locally bounded. Thus we can apply Theorem 4.2.19 of the script to derive that for each $t \geq 0$,

$$\lim_{n\to\infty} \sup_{0\leq s\leq t} \big|(\mathcal{H}^n \cdot X)_s\big| = 0 \quad \text{in probability.}$$

Thus, we obtain the result directly from the definition of $\mathcal{H}^n$.

(c) Let $(\Omega, \mathcal{F}, P)$ be probability space such that there is a random variable $U \sim \mathcal{N}(0, 1)$ which is $\mathcal{F}$-measurable. Fix any $u > 0$ and consider the process

$$X_t := U \, \mathbf{1}_{[u,\infty)}(t), \quad t \geq 0.$$

Moreover, we let $\mathbb{F}$ be the filtration generated by the process X. By construction, X is right-continuous and $\mathbb{F}$-adapted. Assume that X is locally bounded. Then, as $|\Delta X_u| \leq 2\sup_{t\in[0,u]} |X_t|$, we conclude that $|\Delta X_u| \leq C$ P-a.s. for some constant C by the locally boundedness of X. But by definition of X, this means that $|U| \leq C$ P-a.s. which gives a contradiction as $U \sim \mathcal{N}(0, 1)$.

Exercise 9.2 Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions and let $\sigma \leq \tau$ be two stopping times. Moreover, let Z be a bounded, $\mathcal{F}_\sigma$-measurable random variable. The goal of this exercise is to compute the stochastic integral process $\int Z1_{\rrbracket\sigma,\tau\rrbracket}\,dM$ for an integrator $M \in \mathcal{M}^c_{0,\mathrm{loc}}$.

(a) For a (uniformly integrable) right-continuous martingale $X = (X_t)_{t\geq 0}$, show that the process $Z(X^\tau - X^\sigma)$ is again a (uniformly integrable) right-continuous martingale.
Hint: Use Lemma 4.1.19 from the lecture notes to show the assertion first for Z of the form $Z = 1_A$ for some $A \in \mathcal{F}_\sigma$. Then extend the result to general Z.

(b) Let $M, N \in \mathcal{M}^c_{0,\mathrm{loc}}$. Show that

$$\langle Z(M^\tau - M^\sigma), N\rangle = Z\langle M^\tau - M^\sigma, N\rangle = Z(\langle M,N\rangle^\tau - \langle M,N\rangle^\sigma).$$

(c) Let $M \in \mathcal{M}^c_{0,\mathrm{loc}}$ and set $H := Z1_{\rrbracket\sigma,\tau\rrbracket}$. Show that $H \in L^2_{\mathrm{loc}}(M)$ and

$$\int H\,dM = Z(M^\tau - M^\sigma).$$

Conclude that if M is a (uniformly integrable) martingale, then the stochastic integral $\int H\,dM$ is also a (uniformly integrable) martingale.
Remark: The last statement is **not** true for arbitrary bounded $H \in L^2_{\mathrm{loc}}(M)$.

Solution 9.2

(a) Let $X = (X_t)_{t\geq 0}$ be a uniformly integrable, right-continuous martingale. Set $Y := Z(X^\tau - X^\sigma)$ and fix a stopping time ρ. We will show that $E[|Y_\rho|] < \infty$ and $E[Y_\rho] = 0$. The assertion then follows from Lemma 4.1.19 in the lecture notes.

Since X is uniformly integrable, the stopping theorem yields $E[X_\infty|\mathcal{F}_\gamma] = X_\gamma$ for any stopping time γ. In particular, the family $\{X_\gamma : \gamma \text{ a stopping time}\}$ is uniformly integrable (i.e., X is of *class (D)*), hence bounded in L^1. It follows that

$$E[|Y_\rho|] \leq C(E[|X_{\tau\wedge\rho}|] + E[|X_{\sigma\wedge\rho}|]) < \infty$$

where $C > 0$ is any constant bounding Z.

Next, we show that $E[Y_\rho] = 0$. By a monotone class argument or simply measure-theoretic induction, we may assume that $Z = 1_A$ for some $A \in \mathcal{F}_\sigma$. Then $\tau_A := \tau 1_A + \infty 1_{A^c}$ and $\sigma_A := \sigma 1_A + \infty 1_{A^c}$ are stopping times and

$$E[Y_\rho] = E[1_A(X_{\rho\wedge\tau} - X_{\rho\wedge\sigma})] = E[X_{\rho\wedge\tau_A} - X_{\rho\wedge\sigma_A}] = 0,$$

where we use the stopping theorem in the last equality.

If X is not uniformly integrable, then assuming that ρ is bounded, almost the same proof yields that Y is a martingale (but not uniformly integrable in general), c.f. Remark 4.(1.20) in the lecture notes.

(b) The equality $B := Z\langle M^\tau - M^\sigma, N\rangle = Z(\langle M,N\rangle^\tau - \langle M,N\rangle^\sigma)$ follows from bilinearity of $\langle\cdot,\cdot\rangle$ and from the fact that for any stopping time τ,

$$\langle M^\tau, N\rangle = \langle M, N^\tau\rangle = \langle M,N\rangle^\tau.$$

Next, we note that $Y := Z(M^\tau - M^\sigma) \in \mathcal{M}^c_{0,\mathrm{loc}}$ by part **a)** and localisation. So $\langle Y, N\rangle$ is well-defined. We also note that the process B is continuous and of finite variation. Moreover, since $B = 0$ on $\llbracket 0,\sigma\rrbracket$, we can write $B = (Z1_{\rrbracket\sigma,\infty\rrbracket})(\langle M,N\rangle^\tau - \langle M,N\rangle^\sigma)$ to see that B is also adapted.

Setting $X := (M^\tau - M^\sigma)N - \langle M^\tau - M^\sigma, N\rangle \in \mathcal{M}^c_{0,\mathrm{loc}}$ and noting that $X^\sigma = 0$, we have

$$YN - B = Z((M^\tau - M^\sigma)N - \langle M^\tau - M^\sigma, N\rangle) = Z(X - X^\sigma).$$

By part **a)** and localisation, $Z(X - X^\sigma) \in \mathcal{M}^c_{0,\mathrm{loc}}$. Thus, as $\langle Y, N\rangle$ is the unique process $\widetilde{B}$ of cFV_0 such that $MN - \widetilde{B} \in \mathcal{M}^c_{0,loc}$, we conclude by uniqueness that $\langle Y, N\rangle = B$.

(c) Clearly, $H := Z1_{\rrbracket\sigma,\tau\rrbracket}$ is left-continuous. Moreover for $t \geq 0$, the second factor in $H_t = (Z1_{\{\sigma<t\}})1_{\{t\leq\tau\}}$ is $\mathcal{F}_t$-measurable since τ is a stopping time, while the $\mathcal{F}_t$-measurability of the first factor follows from **4-3 c)** via measure-theoretic induction. Thus, H is adapted and hence predictable. Since H is also bounded, it follows that $H \in L^2_{\mathrm{loc}}(M)$.

Now for any $N \in \mathcal{M}^c_{0,\mathrm{loc}}$, we have

$$\langle Z(M^\tau - M^\sigma), N\rangle \overset{\text{b)}}{=} Z(\langle M, N\rangle^\tau - \langle M, N\rangle^\sigma) = \int Z1_{\rrbracket\sigma,\tau\rrbracket}\, d\langle M, N\rangle = \int H\, d\langle M, N\rangle.$$

Thus by the defining property of the stochastic integral, $\int H\, dM = Z(M^\tau - M^\sigma)$.

Finally, from part **a)**, we see that if M is a (uniformly integrable) martingale, then $\int H\, dM = Z(M^\tau - M^\sigma)$ is a (uniformly integrable) martingale.

Exercise 9.3 Let $(B_t)_{t\geq 0}$ be a Brownian motion. Fix any $0 < T < \infty$ and let $f \in L^2([0,T])$ be a deterministic function. For any $0 \leq a < b \leq T$ we set

$$\mathcal{J}_{a,b} := \int_a^b f(s)\, dB_s.$$

Moreover, for any $t \in [0,T]$, we denote $\mathcal{J}_t := \mathcal{J}_{0,t}$.

(a) Show that the process $\mathcal{J} := (\mathcal{J}_{a,b})_{0\leq a\leq b\leq T}$ is a centered Gaussian process and calculate its covariance function.

(b) Show that the process $(\mathcal{J}_t)_{t\in[0,T]}$ has the same law as the process $Y := (Y_t)_{t\in[0,T]}$ defined by

$$Y_t := B_{\int_0^t f^2(s)\, ds}.$$

Solution 9.3

(a) Note that $\cdot .B$ is an isometry of $\mathcal{L}^2$. Thus,

$$\mathbb{E}\left[\mathcal{J}_{a,b}\mathcal{J}_{a',b'}\right] = \mathbf{1}_{\{a\vee a' \leq b\wedge b'\}} \int_{a\vee a'}^{b\wedge b'} f(s)^2 ds$$

Noting that $\mathcal{J}_t$ is a local martingale such that $\mathbb{E}\left[\mathcal{J}_t^2\right] < \infty$ we have that $\mathbb{E}\left[\mathcal{J}_{a,b}\right] = 0$.

To see that it is a Gaussian process we start by taking f to be a continuous function. Using Exercise 9.1 (b) we know that if f is continuous, and if $(\Pi_n)_{n\in\mathbb{N}}$ is a partition with $\lim_{n\to\infty} |\Pi_n| = 0$ we have that

$$\mathcal{J}_{a,b} = \int_a^b f(s) dB_s = \lim_{n\to\infty} \sum_{t_i \leq t, t_i \in \Pi_n n} f(t_i)(B_{t_{i+1}} - B_{t_i}).$$

Thus, thanks to Problem 3.3 if f is continuous $\sum_{i=1}^N \lambda_i \mathcal{J}_{a_i,b_i}$ a normal random variable. Due to the fact that $\cdot .B$ is an isometry we have that if $f_n \overset{\mathcal{L}^2}{\to} f$, then $\mathcal{J}_{a,b}(f_n) \to \mathcal{J}_{a,b}(f)$ in $\mathcal{L}^2(\Omega)$. Thus, due to the fact that continuous function are dense in $\mathcal{L}^2([0,T])$, if $f \in \mathcal{L}^2([0,T])$, there exists f_n continuous functions such that $f_n \overset{\mathcal{L}^2}{\to} f$ thus $\sum_{i=1}^N \lambda_i \mathcal{J}_{a_i,b_i}(f_n) \overset{\mathcal{L}^2}{\to} \sum_{i=1}^N \lambda_i \mathcal{J}_{a_i,b_i}(f)$. Again by Problem 3.3, $\sum_{i=1}^N \lambda_i \mathcal{J}_{a_i,b_i}(f)$ is a normal random variable. Thus, $\mathcal{J}_{a,b}$ is a Gaussian process.

(b) By the previous exercise, we know that for any $t \geq 0$, $E[\mathcal{J}_t] = 0$. For $g(t) := \int_0^t f^2(s)\, ds$, it is clear that the process $Y := (Y_t)_{t\in[0,T]}$ defined by

$$Y_t := B_{g(t)}$$

is a Gaussian process with $E[Y_t] = 0$ for any $t \geq 0$ as Brownian motion is. Moreover, for any $s, t \in [0,T]$, due to the Covariance property of Brownian motion, we obtain that

$$\mathrm{Cov}\big(B_{g(s),g(t)}\big) = \min\big(g(s), g(t)\big) = \mathbb{E}[\mathcal{J}_s \mathcal{J}_t].$$

We conclude that the processes $\mathcal{J}$ and Y are both Gaussian processes with same expectations and Covariance functions, thus have the same finite dimensional marginal distributions.