PART 2 : FK - PERCOLATION

FK-percolation (also called "the random-cluster model") is a percolation process, invented in 1969 by Cees Fortuin and Piet Kasteleyn. We would like to give two main motivations for this 1-parameter family of processes (indexed by $q \geq 1$)

1) it is a generalization of Bernoulli bond percolation (which corresponds to the case when $q = 1$), and it shares many properties with Bernoulli percolation. In particular, several techniques developed for Bernoulli percolation will also apply for FK-percolation. This makes this process very suitable for a mathematical and rigorous analysis.

2) it is related to other models in statistical mechanics. For $q = 2$, the model is related to Ising model: first we will see that it can be seen as a geometric representation of Ising model by expanding the partition function, but in the course we will establish a deeper relation based on the Edwards-Sokal coupling. More generally, for $q \geq 2$ integer, FK percolation is related to Potts model, a generalization of Ising to more than two possible spin-values.

In this second part of the course, we will

- define FK-percolation on a finite subgraph of $\mathbb{Z}^d$ and present basic properties of the model. As for Ising model, we will see that boundary conditions play an important role.

- present the connections with Ising model via the Edwards-Sokal coupling.

- define FK percolation in infinite-volume and define the phase transition.
  This part will be very similar to the corresponding one for the Ising model.

- prove a fundamental result concerning sharpness of the phase transition.

- discuss the planar case (on $\mathbb{Z}^2$). Based on duality we will be able to compute the critical value.

INTRODUCTION

Before defining the model, we describe a new geometric representation for the Ising model with free b.c. and no external magnetic field.

Recall that the Hamiltonian of the free Ising model on $\Lambda \subset\subset \mathbb{Z}^d$ is

$$H^{\emptyset}_{\Lambda,\beta,0}(\sigma) = -\beta \sum_{\{i,j\} \in E_\Lambda} \sigma_i \sigma_j \qquad (\text{for } \sigma \in \Omega_\Lambda)$$

$$= -\beta |E| + 2\beta \sum_{\{i,j\} \in E_\Lambda} \mathbb{1}_{\sigma_i \neq \sigma_j}$$

Therefore

$$Z^{\emptyset}_{\Lambda,\beta,0} = \sum_{\sigma \in \Omega_\Lambda} e^{-H^{\emptyset}_{\Lambda,\beta,0}(\sigma)}$$

$$= \sum_{\sigma \in \Omega_\Lambda} e^{+\beta|E|} \prod_{\{i,j\} \in E_\Lambda} \left(e^{-2\beta} + (1-e^{-2\beta}) \mathbb{1}_{\sigma_i = \sigma_j} \right)$$

$$= e^{+\beta|E|} \sum_{\sigma \in \Omega_\Lambda} \sum_{\omega \in \{0,1\}^{E_\Lambda}} e^{-2\beta c(\omega)} (1-e^{-2\beta})^{o(\omega)} \prod_{ij : \omega(ij)=1} \mathbb{1}_{\sigma_i = \sigma_j}$$

where $o(\omega) = |\{ ij \in E_\Lambda : \omega(ij) = 1\}|$ "open edges in ω"

$c(\omega) = |\{ij \in E_\Lambda : \omega(ij) = 0\}|$ "closed ———"

$$= \sum_{\omega \in \{0,1\}^{E_\Lambda}} e^{-2\beta c(\omega)} (1-e^{-2\beta})^{o(\omega)} \underbrace{\sum_{\sigma \in \Omega_\Lambda} \prod_{ij : \omega(ij)=1} \mathbb{1}_{\sigma_i = \sigma_j}}$$

$$= \begin{cases} 1 & \text{if } \sigma \text{ is constant on each connected component of } \omega. \\ 0 & \text{otherwise} \end{cases}$$

For fixed $\omega \in \{0,1\}^{E_\Lambda}$, the number of spin configurations $\sigma \in \Omega_\Lambda$ that are constant on the connected components of ω is exactly $2^{k(\omega)}$ where $k(\omega)$ is the number of connected components in ω. (There is one possible value, + or −, for each connected component.)

Finally, we get the following expansion of the partition function.

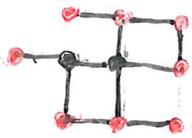
$$\boxed{Z^{\emptyset}_{\Lambda,\beta,0} = e^{+\beta|E|} \cdot \sum_{\omega \in \{0,1\}^{E_\Lambda}} p^{o(\omega)} (1-p)^{\ell(\omega)} 2^{k(\omega)}}$$

where $p = 1 - e^{-2\beta}$

CHAPTER 1 : FK - PERCOLATION ON A FINITE GRAPH.

1 DEFINITIONS

- We call <u>subgraph of $\mathbb{Z}^d$</u> a pair $G = (V, E)$ where $E \subset E_{\mathbb{Z}^d}$ is the set of <u>edges</u> and $V = \bigcup_{e \in E} e$ is the set of <u>vertices</u> induced by E.



a subgraph of $\mathbb{Z}^2$ with 11 edges and 10 vertices. the vertex boundary is represented by the red vertices.

- The <u>vertex boundary</u> of G is defined by

$$\partial G := \{ x \in V : \exists y \in \mathbb{Z}^d \setminus V, \{x, y\} \in E_{\mathbb{Z}^d} \}$$

- A <u>boundary condition</u> (b.c.) is a partition ξ of ∂G.

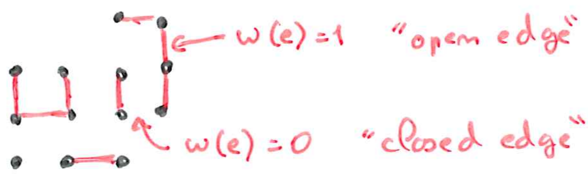
$$\xi = \{ P_1, \dots, P_k \} \quad \text{where} \begin{cases} \partial G = P_1 \cup \dots \cup P_k \\ i \neq j \Rightarrow P_i \cap P_j = \emptyset. \end{cases}$$

<u>Exples</u> the <u>free b.c.</u> $f = \{ \{x\}, x \in \partial G \}$ the <u>wired b.c.</u> $w = \{ \partial G \}$.

- A <u>percolation configuration</u> is an element $\omega = (\omega(e), e \in E) \in \{0,1\}^E$.

One can identify a percolation configuration ω to the subgraph of G given by $(V, \{ e \in E : \omega(e) = 1 \})$.

A graph G



A percolation configuration in G.

We denote by

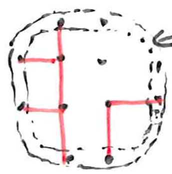
$\sigma(\omega) := |\{e \in E : \omega(e) = 1\}|$ the number of open edges.

$\beta(\omega) := |\{e \in E : \omega(e) = 0\}|$ ———— closed ——

Given a b.c. $\xi = \{P_1, \dots, P_k\}$ and a configuration $\omega \in \{0,1\}^E$ we define $k_\xi(\omega)$ to be the number of clusters in ω where all the clusters intersecting the same P_i count as one.

Example : $G =$  $\omega =$  $\sigma(\omega) = 7$, $\beta(\omega) = 5$

$k_f(\omega) = 5$ $k_w(\omega) = 2$



the element of ∂G are "wired" together

Definition:

Let $p \in [0,1]$, $q > 0$. The FK-percolation measure on G with edge-weight p, cluster-weight q and b.c. ξ, is defined by

$$\boxed{\forall \omega \in \{0,1\}^E \quad \phi^{\xi}_{G,p,q}(\omega) := \frac{p^{\sigma(\omega)}(1-p)^{\beta(\omega)}q^{k_\xi(\omega)}}{Z^{\xi}_{G,p,q}}.}$$

where $Z^{\xi}_{G,p,q} := \sum_{\omega \in \{0,1\}^E} p^{\sigma(\omega)}(1-p)^{\beta(\omega)} q^{k_\xi(\omega)}$

Rks:

1. For $q=1$ we have $\phi^{\xi}_{G,p,q}(\omega) = p^{o(\omega)}(1-p)^{c(\omega)}$ and therefore $\phi^{\xi}_{G,p,1}$ coincides with Bernoulli bond percolation.

2. As q increases, the measure "favors" configurations with more disjoint clusters.

3. Even if we will not consider these objects in this class, we want to mention that the limit as q tends to 0 is related to other important objects in statistical mechanics.

$$\phi^{b}_{p,q} \xrightarrow[q\to 0]{} \begin{cases} \text{"uniform connected subgraph" if } p=\frac{1}{2}. \\ \text{"uniform spanning tree" if } p \searrow 0 \text{ and } q/p \to 0 \\ \text{"uniform spanning forest" if } p=q. \end{cases}$$

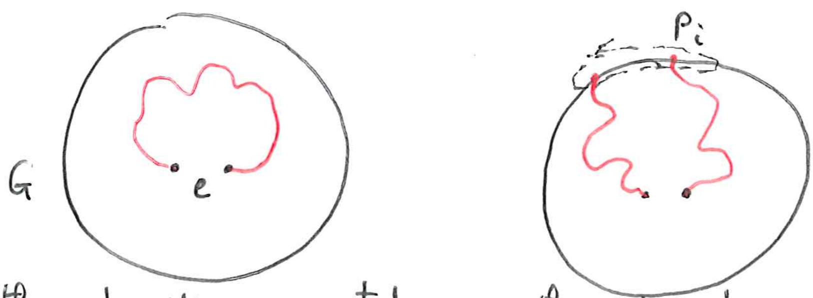
2) MAIN PROPERTIES

Lemma:

Let $e\in E$, $\omega\in\{0,1\}^E$. Then

$$\frac{\phi^{\xi}_{G,p,q}(\omega^e)}{\phi^{\xi}_{G,p,q}(\omega_e)} = \begin{cases} \frac{p}{1-p} & \text{if the extremities of } e \text{ are connected in } (\omega_e,\xi) \\ \frac{p}{q(1-p)} & \text{otherwise.} \end{cases}$$

We say that two vertices are connected in (ω,ξ) if there are connected in ω or they are connected to the same element of the partition ξ.



the extremities connected in ω

the extremities connected to the same element of ξ

Notation: we write $x \leftrightarrow y$ in (ω, ξ) if x and y are connected in (ω,ξ) and simply $x \leftrightarrow y$ in ω if they are connected in ω.

Pf: $(e=\{x,y\})$

$$\frac{\phi^{\xi}_{G,p,q}(\omega^e)}{\phi^{\xi}_{G,p,q}(\omega_e)} = \frac{p}{1-p}\, q^{\underbrace{k^{\xi}(\omega^e) - k^{\xi}(\omega_e)}_{\substack{= -1 \text{ if } x \not\leftrightarrow y \text{ in } (\omega,\xi) \\ = 0 \text{ if } x \leftrightarrow y \text{ in } (\omega,\xi)}}}$$

Rk: Let $\psi \in \{0,1\}^E$, $e = \{x,y\} \in E$

$$\phi^{\xi}_{G,p,q}\left[\omega(e)=1 \mid \forall f \neq e \ \ \omega(f)=\psi(f)\right] = \frac{1}{1+\phi[\psi_e]/\phi[\psi^e]}$$

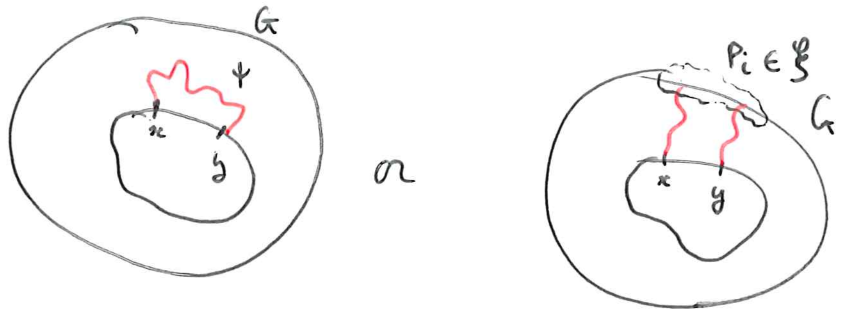
$$= \begin{cases} p & \text{if } x \leftrightarrow y \text{ in } (\omega,\xi) \\ \dfrac{pq}{pq+(1-p)} & \text{otherwise.} \end{cases}$$

Theorem: [DOMAIN MARKOV PROPERTY]

Let $G'=(V',E')$ be a subgraph of $\mathbb{Z}^d$ s.t. $E' \subset E$.
Let $p \in [0,1)$, $q > 0$, ξ b.c. for G.
Then $\forall \psi \in \{0,1\}^{E \setminus E'}$ $\forall \eta \in \{0,1\}^{E'}$

$$\phi^{\xi}_{G,p,q}\left[\omega_{|E'} = \eta \mid \omega_{|E\setminus E'} = \psi\right] = \phi^{\psi^{\xi}}_{G',p,q}[\eta]$$

where ψ^{ξ} is the partition of $\partial G'$ induced by ψ and ξ: $x, y \in \partial G'$ are in the same element of the partition if x is connected to y in (ψ, ξ).



From now and until the end of the course, we always assume

$$\boxed{q \geq 1}$$

Theorem [FKG inequality]

Let $p \in [0,1)$, $\underline{q \geq 1}$. Let ξ b.c.
For every $A, B \subset \{0,1\}^E$ increasing events.

$$\phi^{\xi}_{G,p,q}(A \cap B) \geq \phi^{\xi}_{G,p,q}(A)\, \phi^{\xi}_{G,p,q}(B) \quad [FKG]$$

Proof:

By the lemma, we have for every $\eta \leq \psi$ and every $e = \{x,y\}$

$$\frac{\phi^{\xi}_{G,p,q}(\eta^e)}{\phi^{\xi}_{G,p,q}(\eta_e)} = \frac{p}{1-p}\, q^{1\!\!1[x \leftrightarrow y \text{ in } (\eta,\xi)]}$$

$$\underset{(q \geq 1)}{\leq} \frac{p}{1-p}\, q^{1\!\!1[x \leftrightarrow y \text{ in } (\psi,\xi)]}$$

$$\leq \frac{\phi^{\xi}_{G,p,q}(\psi^e)}{\phi^{\xi}_{G,p,q}(\psi_e)}$$

Holley criterion for FKG inequality concludes the proof. $\square$.

Theorem [COMPARISON BETWEEN B.C.] Let $p \in [0,1]$ $q \geq 1$.

Let $\xi \leq \xi'$ two b.c (meaning that any element of the partition ξ is included in an element of ξ'). Then

$$\phi^{\xi}_{G,p,q} \ll \phi^{\xi'}_{G,p,q}$$

Rk: In particular, we have for every b.c. ξ,

$$\phi^{f}_{G,p,q} \ll \phi^{\xi}_{G,p,q} \ll \phi^{w}_{G,p,q}.$$

Proof: As above, if $\eta \leq \psi$, we can use that

$$\mathbb{1}[x \longleftrightarrow y \text{ in } (\eta,\xi)] \leq \mathbb{1}[x \longleftrightarrow y \text{ in } (\psi,\xi')]$$

to show

$$\frac{\phi^{\xi}_{G,p,q}[\eta^e]}{\phi^{\xi}_{G,p,q}[\eta_e]} \leq \frac{\phi^{\xi'}_{G,p,q}[\psi^e]}{\phi^{\xi'}_{G,p,q}[\psi_e]},$$

and then apply the Holley criterion.

Theorem: [MONOTONICITY PROPERTIES]

Let ξ be a b.c..

- If $p \leq p'$ and $q \geq q' \geq 1$, then $\phi^{\xi}_{G,p,q} \ll \phi^{\xi}_{G,p',q'}$

  ($(p,q) \to \phi^{\xi}_{G,p,q}$ is increasing in p, decreasing in q)

- If $1 \leq q \leq q'$ and $\dfrac{p}{q(1-p)} \leq \dfrac{p'}{q'(1-p')}$, then $\phi^{\xi}_{G,p,q} \ll \phi^{\xi}_{G,p',q'}$

  (even if $q \leq q'$, taking p' sufficiently larger than p' gives rise to the domination $\phi_{p,q} \ll \phi_{p',q'}$)

Proof: . To prove the two results it suffices to prove that for every $p \leq p'$, and $q, q' \geq 1$,

$$\frac{p}{q(1-p)} \leq \frac{p'}{q'(1-p')} \Rightarrow \phi^{\xi}_{G,p,q} \ll \phi^{\xi}_{G,p',q'}$$

Again, we use the Holley criterion. The inequalities $\frac{p}{q(1-p)} \leq \frac{p'}{q'(1-p')}$ and $p \leq p'$ imply for all $a, b \in \{0,1\}$ s.t. $a \leq b$,

$$\frac{p}{q(1-p)} q^{a} \leq \frac{p'}{q'(1-p')} q'^{b}$$

(distinguish between the 3 possible cases)

Therefore $\forall \eta \leq \psi$,

$$\frac{p}{q(1-p)} q^{\mathbb{1}[x \leftrightarrow y \text{ in } (\eta, \xi)]} \leq \frac{p'}{(1-p')q'} q'^{\mathbb{1}[x \leftrightarrow y \text{ in } (\psi, \xi)]}$$ □

Applications $\forall e \in E$

1. $\phi^{\xi}_{G,p,q}[\omega(e)=1]$ is $\nearrow$ in p and decreasing in q.

2. $\underbrace{\phi_{G,p',1}}_{\text{Bernoulli}(p')} \ll \phi^{\xi}_{G,p,q} \ll \underbrace{\phi_{G,p,1}}_{\text{Bernoulli}(p)}$ for $p' = \frac{p}{p+q(1-p)}$

In

CHAPTER 2 $\boxed{\text{EDWARDS - SOKAL COUPLING}}$

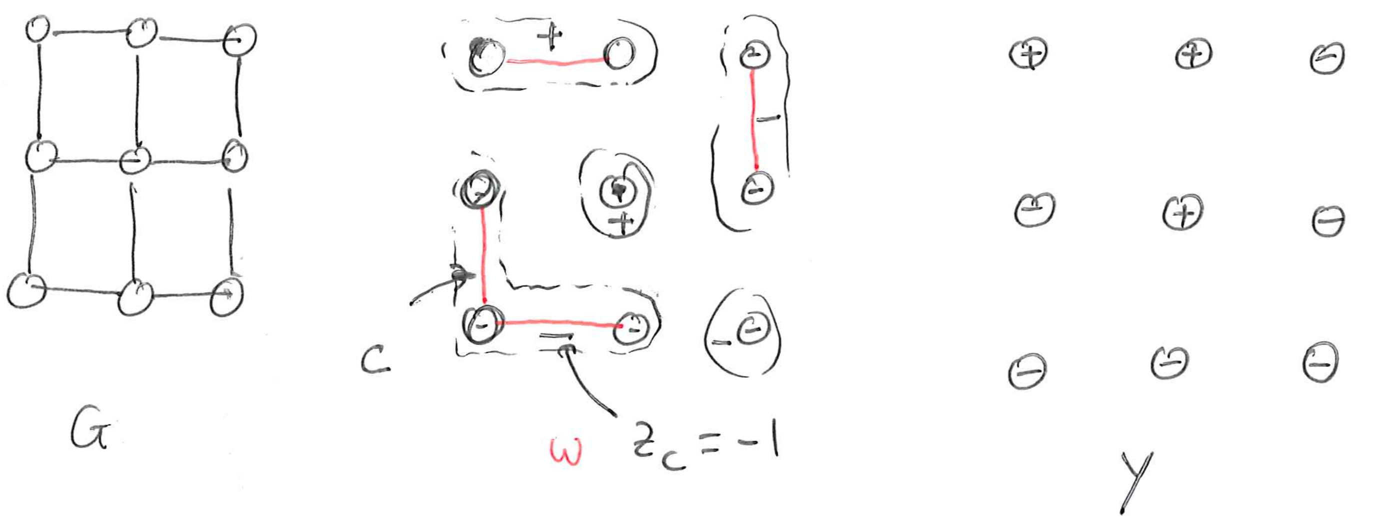
1) $\boxed{\text{Free Measures}}$

$\boxed{q=2}$ ($q\in\{2,3,\dots\}$ FK $\leftrightarrow$ ~~Potts~~ q-state potts model)

Given $\Lambda\subset\subset\mathbb{Z}^d$, we consider the graph $G=(\Lambda, E_\Lambda)$.

Let $p\in[0,1)$, let $\omega\in\{0,1\}^{E_\Lambda}$

- To each cluster C of ω, associate a random variable Z_C s.t. such that:
  - (Z_C) are independent
  - $\mathbb{P}(Z_C=+1)=\frac{1}{2}$



- Define $Y\in\Omega_\Lambda$ by setting for every cluster C and every $i\in C$

$$Y(i)=Z_C$$

We say that Y is <u>coloring of the clusters of</u> ω.

<u>Thm</u>: [Coupling between ϕ^β and $\mu^\emptyset$]

Let $p\in[0,1]$ and $\beta\geq 0$ such that $p=1-e^{-2\beta}$.

If $X\sim\phi^{\beta}_{G,p,2}$ then an independent coloring Y of ω ~~is an~~ has the law $\mu^\emptyset_{\Lambda,\beta,0}$

Let $\sigma \in \Omega_\Lambda$, Let $D_\sigma = \{\{i,j\} \in E : \sigma_i \neq \sigma_j\}$.

Define $X' \in \{0,1\}^E$ by

$$P[X(e)=1] = \begin{cases} p & \text{if } e \notin D_\sigma \\ 0 & \text{if } e \in D_\sigma \end{cases}$$

in such a way that $(X(e))$ are independent.

We say that X' is a p-percolation on $E \setminus D_\sigma$.

Thm: Let $\beta \geq 0$ and $p = 1 - e^{-2\beta}$

① If $X \sim \phi^{f}_{G,p,2}$, and Y is an ind. coloring of X, then $Y \sim \mu^{\emptyset}_{\Lambda,\beta,0}$

② If $Y' \sim \mu^{\emptyset}_{\Lambda,\beta,0}$, and X' is a p-percolation in $E \setminus D_{Y'}$, then $X' \sim \phi^{f}_{G,p,2}$.

Remark: We will actually prove that (X,Y) and (X',Y') have the same law. This common law is a coupling of $\phi^{f}_{G,p,2}$ and $\mu^{\emptyset}_{\Lambda,\beta}$.

(demoted by $\sigma\sim\omega$) ③

Proof: ~~Def~~ We say σ is compatible with ω if ~~x~~, for every $x,y\in\Lambda$

$$(x \longleftrightarrow y \text{ in } \omega) \Rightarrow (\sigma_x = \sigma_y)$$

(in other words σ is compatible with ω iff it is constant on its connected components.

$$= \mathbb{P}[X=\omega] \cdot \mathbb{P}[Y=\sigma \mid X=\omega]$$

$$\mathbb{P}[X=\omega, Y=\sigma] = \frac{1}{Z^{f}_{G,p,2}} p^{\sigma(\omega)} (1-p)^{c(\omega)} 2^{k(\omega)} \times \mathbb{1}_{\omega\sim\sigma} 2^{-k(\omega)}$$

$$= \frac{1}{Z^{f}_{G,p,2}} \mathbb{1}_{\sigma\sim\omega} p^{\sigma(\omega)} (1-p)^{c(\omega)}.$$

$$= \mathbb{P}[Y'=\sigma] \cdot \mathbb{P}(X'=\omega \mid Y'=\sigma)$$

$$\mathbb{P}[X'=\omega, Y'=\sigma] = \mathbb{1}_{\sigma\sim\omega} \frac{1}{Z^{\emptyset}_{\Lambda,\beta,0}} e^{+\beta|E| - 2\beta|D_\sigma|} \times p^{o(\omega)} \underbrace{(1-p)^{c(\omega)-|D_\sigma|}}_{=(1-p)^{c(\omega)} e^{2\beta|D_\sigma|}}$$

$$= \mathbb{1}_{\sigma\sim\omega} \frac{e^{+\beta|E|}}{Z^{\emptyset}_{\Lambda,\beta,0}} p^{\sigma(\omega)} (1-p)^{c(\omega)}$$

By summing the two equations above over σ,ω we get

i.e. $$\frac{1}{Z^{f}_{G,p,2}} \sum_{\sigma,\omega} \mathbb{1}_{\sigma\sim\omega} p^{\sigma(\omega)} (1-p)^{c(\omega)} = 1 = \frac{e^{\beta|E|}}{Z^{\emptyset}_{\Lambda,\beta,0}} \sum_{\sigma,\omega} \mathbb{1}_{\sigma\sim\omega} p^{\sigma(\omega)} (1-p)^{c(\omega)}$$

we recover the result $$\frac{1}{Z^{f}_{G,p,2}} = \frac{e^{\beta|E|}}{Z^{\emptyset}_{\Lambda,\beta,0}}.$$

Therefore, we obtain that for every $(\omega,\sigma) \in \{0,1\}^E \times \Omega_\Lambda$

$$\mathbb{P}[X=\omega, Y=\sigma] = \mathbb{P}[X'=\omega, Y'=\sigma]$$

In particular
- Y has the same law as Y', ie $Y \sim \mu^{\emptyset}_{\Lambda,\beta}$
- X' has the same law as X, ie $X' \sim \phi^{b}_{G,p,2}$ ■

Rk: We have proved that (X,Y) and (X',Y') have the same law P on $\{0,1\}^E \times \Omega_\Lambda$ given by $P[(\omega,\sigma)] = \frac{1}{Z} \mathbb{1}_{\omega \sim \sigma}$

P is a coupling of ϕ_G and μ_Λ

Proposition:

> Let $\beta \geq 0$ and $p = 1-e^{-2\beta}$. Then, for every $i,j \in \Lambda$
> We have $\langle \sigma_i \sigma_j \rangle^{\emptyset}_{\Lambda,\beta} = \phi^{b}_{G,p,2}[i \leftrightarrow j \text{ in } \omega]$

Rk: This implies directly that

$$\langle \sigma_i, \sigma_j \rangle^{\emptyset}_{\Lambda,\beta} = \phi^{b}_{G,1-e^{-2\beta},2}(i \leftrightarrow j \text{ in } \omega)$$

Since the event $i \leftrightarrow j$ is increasing.

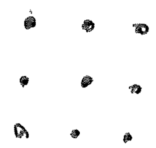
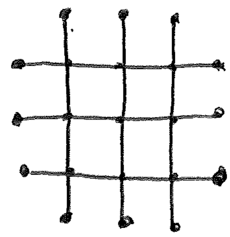
Proof:

Let $X \sim \phi^{b}_{G,p,2}$, let Y be a random coloring of the clusters of X. In particular $Y \sim \mu^{\emptyset}_{\Lambda,\beta}$ and we have

$$\begin{aligned}
\langle \sigma_i \sigma_j \rangle^{\emptyset}_{\Lambda,\beta} &= \mathbb{E}[Y_i Y_j] \\
&= \underbrace{\mathbb{E}[Y_i Y_j \mid i \leftrightarrow j \text{ in } X]}_{=1} \mathbb{P}[i \leftrightarrow j \text{ in } X] \\
&\quad + \underbrace{\mathbb{E}[Y_i Y_j \mid i \not\leftrightarrow j \text{ in } X]}_{=0} \mathbb{P}[i \not\leftrightarrow j \text{ in } X] \\
&= \mathbb{P}[i \leftrightarrow j \text{ in } X] = \phi^{b}_{G,p,2}[i \leftrightarrow j \text{ in } \omega]
\end{aligned}$$

2) Coupling wired FK / + Ising and wired FK / − Ising.

Let $\Lambda \subset\subset \mathbb{Z}^d$. We consider the graph $\bar{G}_\Lambda = (\bar{V}, \bar{E}) = (V, \bar{E}_\Lambda)$ induced by the edges of $\bar{E}_\Lambda$ (V contains Λ and the vertices at the external boundary of Λ)

If $\Lambda = \Lambda_1 =$  Then $\bar{G} =$ 

Rk: an Ising configuration $\sigma \in \Omega_\Lambda^+$ can be identified to an element $\sigma \in \{-1,1\}^V$ with $\sigma = 0$ on $\partial \bar{G}$

Thm: Let $\beta \geq 0$ and $p = 1 - e^{-2\beta}$.

1) Let $X \sim \phi^w_{\bar{G},p}$, let Y be a random coloring of the clusters of X, with the additionnal condition that all the clusters touching ∂G are colored $+1$ (resp. -1) then $Y \sim \mu^+_{\Lambda,\beta}$ (resp. $Y \sim \mu^-_{\Lambda,\beta}$).

2) Let $Y' \sim \mu^+_{\Lambda,\beta}$, let X' be a p-percolation in $E \setminus D_{Y'}$. Then $X' \sim \phi^w_{\bar{G},p,2}$

Proof: Let $\omega \in \{0,1\}^{\bar{E}}$, $\sigma \in \{-1,1\}^V$ where $\sigma(i) = +1 \ \forall i \in \partial G$.

$$\mathbb{P}[X = \omega, Y = \sigma] = \frac{1}{Z^b_{\bar{G},p,2}} \mathbb{1}_{\omega \sim \sigma}\, p^{o(\omega)} (1-p)^{c(\omega)}\, 2^{k_w(\omega)} \times \frac{1}{2^{k_w(\omega)-1}}$$

$$= \frac{2}{Z^b_{\bar{G},p,2}} \mathbb{1}_{\omega \sim \sigma}\, p^{o(\omega)} (1-p)^{c(\omega)}$$

$$\mathbb{P}[X'=\omega, Y'=\sigma] = \frac{1}{Z^+_{\Lambda,\beta}} e^{+\beta|E|} \times e^{-2\beta|D_{\sigma}|} \times \mathbb{1}_{\sigma\sim\omega}\, p^{\sigma(\omega)} (1-p)^{c(\omega)-|D_\sigma|}$$

$$= \frac{e^{\beta|E|}}{Z^+_{\Lambda,\beta}} \mathbb{1}_{\sigma\sim\omega}\, p^{\sigma(\omega)} (1-p)^{c(\omega)}.$$

Proposition:

Let $\beta \geq 0$ and $p = 1-e^{-2\beta}$. Then, if $0 \in \Lambda$

$$\langle \sigma_0 \rangle^+_{\Lambda,\beta} = \phi^w_{\bar{G},p}[0 \leftrightarrow \partial G \text{ in } \omega]$$

Proof: Let X, Y be two random variables as in the theorem

$$\langle \sigma_0 \rangle^+_{\Lambda,\beta} = \mathbb{E}[Y_0]$$

$$= \mathbb{E}[\underbrace{Y_0}_{=1} \mid 0 \leftrightarrow \partial \bar{G} \text{ in } X]\, \mathbb{P}[0 \leftrightarrow \partial \bar{G} \text{ in } X]$$

$$+ \underbrace{\mathbb{E}[Y_0 \mid 0 \not\leftrightarrow \partial \bar{G} \text{ in } X]}_{=0}\, \mathbb{P}[0 \leftrightarrow \partial \bar{G} \text{ in } X]$$

$$= \mathbb{P}[0 \leftrightarrow \partial \bar{G} \text{ in } X]$$

$$= \phi^w_{\bar{G},p}[0 \leftrightarrow \partial G \text{ in } \omega]. \qquad \square$$

Proof of the GKS inequalities for $h=0$

Thm: Let $\# \in \{+, \emptyset\}$ then $\forall A, B \subset \Lambda \subset\subset \mathbb{Z}^d$, we have

1) $\langle \sigma_A \rangle^{\#}_{\Lambda,\beta} \geq 0$
2) $\langle \sigma_A \sigma_B \rangle^{\#}_{\Lambda,\beta} \geq \langle \sigma_A \rangle^{\#}_{\Lambda,\beta} \langle \sigma_B \rangle^{\#}_{\Lambda,\beta}$.

We prove here these inequalities for $\# = \emptyset$.

$\Lambda \subset\subset \mathbb{Z}^d$ $\quad G = (V, E) = (\Lambda, E_\Lambda)$

Lemma: Let $A \subset \Lambda$.

Consider the percolation event.

$$\mathcal{F}_A = \{ \omega \in \{0,1\}^E : \forall C \in \mathcal{C}(\omega) \quad |C \cap A| \text{ is even} \}$$

where $\mathcal{C}(\omega) = \{\text{clusters of } \omega\}$.

Then

$$\langle \sigma_A \rangle^{+}_{\Lambda,\beta} = \phi^{\emptyset}_{G,p} (\mathcal{F}_A) \qquad (p = 1 - e^{-2\beta})$$

Proof:

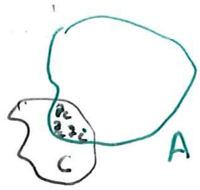
Let $X \sim \phi_{G,p}$ and Y a coloring of the cluster of X.

$$\langle \sigma_A \rangle^{\emptyset}_{\Lambda,\beta} = \mathbb{E}[Y_A]$$

$$= \sum_{\omega \in \{0,1\}^E} \mathbb{E}[Y_A \mid X = \omega] \ \mathbb{P}[X = \omega]$$

Assume $X = \omega$

$$Y_A = \prod_{C \in \mathcal{C}(\omega)} Y_{C \cap A} = \prod_{C \in \mathcal{C}(\omega)} z_C^{|C \cap A|}$$



$$E[Y_A \mid X = \omega] = E\Big[\prod_{C \in \mathcal{C}(\omega)} z_C\Big] = \prod_{C \in \mathcal{C}(\omega)} E[z_C^{|C \cap A|}]$$

$$= \begin{cases} 1 & \text{if } \forall\, C \in \mathcal{C}(\omega) \quad C \cap A \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\langle \sigma_A \rangle^{\emptyset}_{\Lambda,\beta} = \sum_{\substack{\omega \in \{0,1\}^{\omega}}} \mathbb{1}_{\omega \in \mathcal{F}_A}\, \mathbb{P}[X = \omega]$

$$= \mathbb{P}[X \in \mathcal{F}_A]$$

$$= \phi^{f}_{G,p}[\mathcal{F}_A]. \qquad \square$$

<u>Proof of The Theorem for: $\# = \emptyset$, A, B disjoint</u>

1) $\langle \sigma_{\Lambda,\beta} \rangle^{\emptyset}_{\Lambda,\beta} = \phi^{f}_{G,p}[\mathcal{F}_A] \geq 0$

$$\langle \sigma_A \sigma_B \rangle^{\emptyset} = \langle \sigma_{A \cup B} \rangle^{0}$$

$$= \phi_G (\mathcal{F}_{A \cup B})$$

$$\geq \phi (\mathcal{F}_A \cap \mathcal{F}_B) \quad (\text{becaus } \mathcal{F}_A \cap \mathcal{F}_B \subset \mathcal{F}_{A \cup B})$$

$$\overset{\text{FKG}}{\geq} \phi(\mathcal{F}_A)\, \phi(\mathcal{F}_B) \quad (\text{because } \mathcal{F}_A, \mathcal{F}_B \uparrow)$$

$$= \langle \sigma_A \rangle^{\emptyset} \langle \sigma_B \rangle^{\emptyset}.$$

CHAPTER 3: INFINITE VOLUME FK-PERCOLATION

$q \geq 1$ $\quad p \in [0,1)$ fixed.

1) THE WIRED AND FREE INFINITE-VOLUME MEASURES.

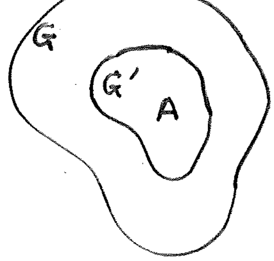
- $\Omega = \{0,1\}^{\mathbb{E}^d}$ $\quad$ ($\mathbb{E}^d$ edges of $\mathbb{Z}^d$)
- For $E \subset\subset \mathbb{E}^d$ $\quad \mathcal{F}_E$ = sigma-algebra generated by $(\omega(e))_{e \in E}$.
- (Rk: if $A \in \mathcal{F}_E$ and $G = (V,E)$ is the graph generated by E, then $\phi^{\xi}_{G,p,q}(A)$ is well defined.)

An event A is <u>local</u> if $\exists E \subset\subset \mathbb{E}^d$ s.t. $A \in \mathcal{F}_E$.

Lemma ("Pushing b.c.") Let $G = (V,E)$ and $G' = (V',E')$ subgraphs s.t. $E' \subset E \subset\subset \mathbb{E}^d$. For every $A \in \mathcal{F}_{E'}$ increasing.

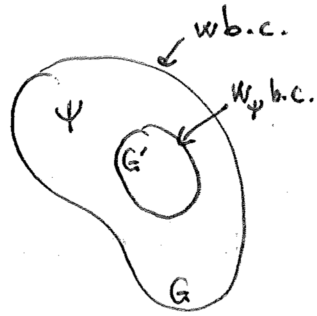
$$\phi^{w}_{G',p,q}(A) \geq \phi^{w}_{G,p,q}(A)$$

$$\phi^{f}_{G',p,q}(A) \leq \phi^{w}_{G,p,q}(A).$$



Pf: $$\phi^{w}_{G,p,q}(A) = \sum_{\psi \in \{0,1\}^{E \setminus E'}} \underbrace{\phi^{w}_{G,p,q}(A \mid \omega_{E \setminus E'} = \psi)} \; \phi^{w}_{G,p,q}(\omega_{E \setminus E'} = \psi)$$

$$\overset{\text{DMP}}{=} \phi^{w_\psi}_{G',p,q}(A) \overset{\text{CBC}}{\leq} \phi^{w}_{G',p,q}(A)$$



$$\leq \phi^{w}_{G',p,q}(A) \sum_{\psi \in \{0,1\}^{E \setminus E'}} \phi^{w}_{G,p,q}(\omega_{E \setminus E'} = \psi) \qquad \square$$

Theorem: Let $p \in [0,1]$, $q \geq 1$. There exist two probability measures $\phi^b_{p,q}$ and $\phi^w_{p,q}$ on Ω characterized by

$\forall A$ local event, $\forall G_n = (V_n, E_n)$ with $E_n \uparrow \mathbb{Z}^d$.

(i) $\phi^w_{p,q}(A) = \lim_{n \to \infty} \phi^w_{G_n,p,q}(A)$,

(ii) $\phi^b_{p,q}(A) = \lim_{n \to \infty} \phi^b_{G_n,p,q}(A)$.

Proof: Let A be a local increasing event

Then for n large enough $\phi^w_{G_{n+1},p,q}(A) \leq \phi^w_{G_n,p,q}(A)$

therefore $\phi^w_{p,q}(A) := \lim_{n \to \infty} \phi^w_{G_n,p,q}(A)$ is well defined

As for Ising, we first extend $\phi^w_{p,q}$ to local events and then to every events by Kolmogorov's theorem. $\square$

Exercices.

1. Prove that $\phi^b_{p,q} \ll \phi^w_{p,q}$.

2. Prove that $\phi^b_{p,q}$ and $\phi^w_{p,q}$ are translation invariant.

3. (difficult) Prove that $\phi^b_{p,q}$ and $\phi^w_{p,q}$ are ergodic: if A is a translation invariant event then

$$\phi^b_{p,q}(A), \phi^w_{p,q}(A) \in \{0,1\}.$$

2 UNIQUENESS OF THE INFINITE VOLUME MEASURE.

Definition:

Let $q \geq 1$, $p \in [0,1]$. We say that there exists a unique infinite-volume FK-measure if $\phi^f_{p,q} = \phi^w_{p,q}$

Remark:

As for Ising model, one can define general infinite-volume FK-measures, and any such measure ϕ would satisfy

$$\phi^f_{p,q} \ll \phi \ll \phi^w_{p,q}.$$

Remark: The FK-percolation measure in finite volume can be rewritten as

$$\phi^{\xi}_{G,p,q}(\omega) = \frac{1}{Z^{\xi}_{G,p,q}} q^{k_{\xi}(\omega)} p^{o(\omega)} (1-p)^{|E|-o(\omega)}$$

$$= \frac{1}{\tilde{Z}} q^{k_{\xi}(\omega)} \cdot \left(\frac{p}{1-p}\right)^{\sum_{e\in E} \omega_e}$$

Here, the parameter p plays a "similar role" as the external field h for Ising.

In particular, the techniques which provides characterizations of uniqueness for Ising in terms of analytic properties in h apply. Here uniqueness can be understood via analytic properties in p.

Theorem: (Characterization of uniqueness)

Fix $q \geq 1$ and $p \in [0,1]$. The following are equivalent

(i) $\phi^{w}_{p,q} = \phi^{b}_{p,q}$

(ii) $\phi^{w}_{p,q}(\omega_e = 1) = \phi^{b}_{p,q}(\omega_e = 1)$

(iii) $p' \longmapsto \phi^{w}_{p',q}(\omega_e = 1)$ is continuous at p.

Proof: Similar to Ising (exercise).

Corollary

Fix $q \geq 1$. The set $\mathcal{D}_q := \{p \in [0,1] : \phi^{w}_{p,q} \neq \phi^{b}_{p,q}\}$ is at most countable.

Proof: $p' \mapsto \phi^{w}_{p',q}(\omega(e) = 1)$ is increasing, hence it has at most countably many discontinuity points. □

Theorem: (sufficient condition for uniqueness.)

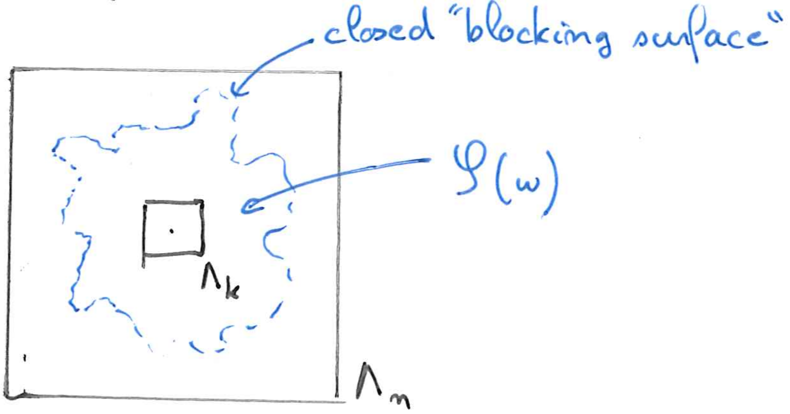
Let $0 \leftrightarrow \infty$ be the event that the component of 0 is infinite. If $\phi^{w}_{p,q}(0 \longleftrightarrow \infty) = 0$ then $\phi^{b}_{p,q} = \phi^{w}_{p,q}$.

Proof: Let A be a local increasing event. Let $k \geq 1$ such that A is measurable with respect to the configuration in $B_k = \{-k, \dots, k\}^d$. We have, for every $n \geq k$,

$$(*) \quad \phi^{b}_{B_n,p,q}(A) \underset{\substack{\uparrow \\ \text{stochastic} \\ \text{domination}}}{\leq} \phi^{w}_{B_n,p,q}(A) \leq \phi^{w}_{B_n,p,q}(A \mid B_k \not\leftrightarrow \partial B_n) + \phi^{w}_{B_n}(B_k \longleftrightarrow \partial B_n)$$

The general idea of the proof is that the event $G_k \not\leftrightarrow \partial G_m$ implies the existence of a "blocking surface" between G_k and ∂G_m. This blocking surface create then free boundary conditions for the configuration inside. Define the random set

$$\mathcal{S}(\omega) = \{ x \in \Lambda_m : x \not\leftrightarrow \partial G_m \text{ in } \omega \}$$



Notice that
- $\Lambda_k \subset \mathcal{S}(\omega)$ if $\Lambda_k \not\leftrightarrow \partial G_m$ in ω.
- the event $\mathcal{S}(\omega) = S$ depends only on the configuration in $\Lambda_m \setminus S$.

We have
$$\phi^w_{B_m,p,q}(A \cap B_k \not\leftrightarrow B_m) = \sum_{B_k \subset S \subset B_m} \underbrace{\phi^w_{B_m,p,q}(A \mid \mathcal{S}=S)}\, \phi^w_{B_m,p,q}(\mathcal{S}=S)$$

$$\overset{MP}{=} \phi^f_{S,p,q}(A)$$

$$\overset{\text{Pushing}}{\leq} \phi^f_{\Lambda_m,p,q}(A)$$

$$\leq \phi^f_{\Lambda_m,p,q}(A) \times \sum_{B_k \subset S \subset B_m} \phi^w_{B_m,p,q}(\mathcal{S}=S)$$

Hence $\phi^w_{B_m,p,q}(A \mid B_k \not\leftrightarrow B_m) \leq \phi^f_{B_m,p,q}(A)$

Plugging this in $(*)$ and letting n tend to infinity concludes the proof.

3. PHASE TRANSITION OF FK-PERCOLATION

As seen in the previous chapter, long-distance correlations for Ising was encoded by the existence of long path for the corresponding FK-percolation process. With this in mind, it is natural to define the phase transition of FK-percolation by looking at the existence of infinite paths. Before giving the definition of the critical parameter, we give a useful expression for the probability of an infinite path from 0.

Lemma: Let $q \geq 1$, $p \in [0,1)$. For every $G_n \uparrow \mathbb{Z}^d$

$$\phi^w_{p,q}(0 \longleftrightarrow \infty) = \lim_{n \to \infty} \phi^w_{G_n,p,q}(0 \longleftrightarrow \partial G_n)$$

Proof: By "pushing the b.c.", we have for every $k \geq n$

$$\phi^w_{G_k,p,q}(0 \longleftrightarrow \partial G_n) \leq \phi^w_{G_n,p,q}(0 \longleftrightarrow \partial G_n)$$

Taking first the limit as k tends to infinity, and then the liminf as n tends to infinity, we obtain

$$\phi^w_{p,q}(0 \longleftrightarrow \infty) \leq \liminf_{n \to \infty} \phi^w_{G_n,p,q}(0 \longleftrightarrow \partial G_n).$$

On the other hand, we trivially have for $k \leq n$

$$\phi^w_{G_n,p,q}(0 \longleftrightarrow \partial G_k) \geq \phi^w_{G_n,p,q}(0 \longleftrightarrow \partial G_n)$$

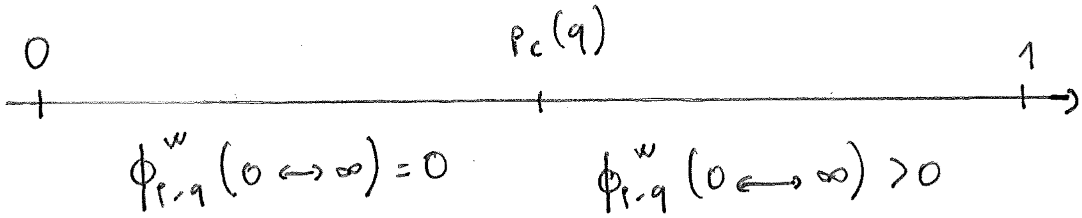
Taking first the limsup as n tends to infinity and then the limit as k tends to infinity, we get.

$$\phi^{w}_{p,q}(0 \longleftrightarrow \infty) \geq \limsup_{n\to\infty} \phi^{w}_{G_n,p,q}(0 \longleftrightarrow \partial G_n) \qquad \blacksquare$$

The expression above directly implies that for fixed $q \geq 1$, $\phi^{w}_{p,q}(0 \longleftrightarrow \infty)$ is increasing in p (as a simple limit of increasing functions -

<u>Definition</u>: Fix $q \geq 1$, define the critical parameter for FK - percolation by.

$$\boxed{p_c(q) := \sup\left\{p \in [0,1] : \phi^{w}_{p,q}(0 \longleftrightarrow \infty) = 0\right\}}$$



<u>Remark</u>: The choice of the wired measure is arbitrary here and the critical parameter does not depend on this choice. Indeed, using that the infinite-volume measure is unique for $p < p_c(q)$ and the almost every-where uniqueness for $p > p_c(q)$, one can prove that

$$\phi^{f}_{p,q}(0 \longleftrightarrow \infty) \begin{cases} = 0 & \text{if } p < p_c(q) \\ > 0 & \text{if } p > p_c(q). \end{cases}$$

Question: What is the monotonicity of $p_c(q)$, as a function of the cluster-weight q ?

4) IMPORTANT RESULTS AND CONJECTURES : AN OVERVIEW.

41) Subcritical phase. ($p < p_c(q)$)

Thm: Fix $q \geq 1$, $p < p_c(q)$. Then,

(i) [UNIQUENESS] $\phi^{w}_{p,q} = \phi^{f}_{p,q}$

(ii) [EXPONENTIAL DECAY] there exists $c > 0$ s.t. for every $m \geq 1$

$$\boxed{\phi^{w}_{G_m,p,q}(0 \longleftrightarrow \partial G_m) \leq e^{-cm}}$$

where $G_m = [-m, m]^2$

(i) follows from the Lemma in Section 2.

(ii) is a recent result. Using tools from the theory of randomized algorithm, it is possible to prove that the function $\theta_m(p) = \phi^{w}_{G_m,p,q}(0 \longleftrightarrow \partial G_m)$ satisfies

(*) $$\theta_m' \geq c \cdot \frac{m}{S_m} \theta_m$$ where $c > 0$ constant and $S_m = \sum_{k=1}^{m} \theta_k$

and this differential inequality implies the result.

4.2 Supercritical phase ($p > p_c(q)$)

Theorem: Fix $q \geq 1$.

(i) [MEAN-FIELD LOWER BOUND] $\exists c > 0$ s.t. $\forall p > p_c(q)$

$$\phi^{w}_{p,q}(0 \longleftrightarrow \infty) \geq c\,(p - p_c(q))$$

(ii) [UNIQUENESS OF THE INFINITE CLUSTER]

Let $\# \in \{f, w\}$. $p > p_c(q)$ then

$$\phi^{\#}_{p,q}[\exists \text{ a unique infinite cluster in } w] = 1.$$

(i) is proved using the differential inequality (*) in the previous page.

To prove (ii) one first need to show that the two measures $\phi^{\#}_{p,q}$, $\# \in \{f, w\}$ are ergodic (translation invariant events satisfy a 0-1 law). Then, the Burton-Keane argument (trifurcation argument) applies, exactly as for Bernoulli percolation.

Conjecture:

If $p > p_c(q)$ then $\phi^{w}_{p,q} = \phi^{f}_{p,q}$

As we will see later, the conjecture can be proved in dimension 2, by duality. Let us here present an argument that suggests why the conjecture should also be true in higher dimension.

In dim. 2, we can prove the following statement.

$$\phi^{b}_{p,q}(0 \longleftrightarrow \infty) > 0 \Longrightarrow \phi^{b}_{p,q} = \phi^{w}_{p,q}.$$

This implication is very similar to the lemma in Section 2. It clearly implies the conjecture since $\phi^{b}_{p,q}(0 \leftrightarrow \infty) > 0$ $\forall p > p_c(q)$. The general idea behind the implication above is the following: If under $\phi^{b}_{p,q}$, there exists an infinite cluster, then this infinite cluster should "create" wired boundary conditions around a big box Λ_m, and we expect that

$$\phi^{b}_{p,q}\left[\,\cdot \;\middle|\; \begin{array}{c}\exists!\ \text{infinite cluster} \\ \text{outside } \Lambda_m\end{array}\right] \simeq \phi^{w}_{\Lambda_m,p,q}[\,\cdot\,)$$

since the conditioning occurs almost surely this would imply

$$\phi^{b}_{p,q} = \phi^{w}_{p,q}$$

This argument can be made rigorous in dimension 2 because the existence of an unique infinite cluster under $\phi^{b}_{p,q}$ implies the existence of a circuit around the origin. Conditioning under such circuit, the configuration inside dominates a wired FK-percolation.

4.3 Critical phase.

The critical behaviour of FK percolation is not well understood, except in few cases.

Thm: In dimension 2, we have.

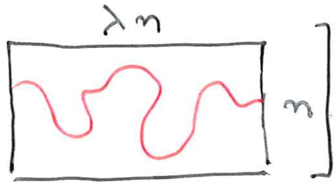
- if $q \leq 4$ $\quad \phi^{w}_{p_c(q),q} = \phi^{f}_{p_c(q),q}$ and $\phi^{w}_{p_c(q)}(0 \leftrightarrow \infty) > 0$

- if $q > 4$ $\quad \begin{cases} \phi^{w}_{p_c(q),q}(0 \leftrightarrow \infty) > 0 \\ \phi^{f}_{p_c(q),q}(0 \leftrightarrow \partial\Lambda_n) \leq e^{-cn}. \end{cases}$

If $q \leq 4$, the model is conjectured to converge to a scaling limit related to SLE processes. This has been proved for $q = 2$ only. In particular the following convergence is expected to hold.

Conjecture $\forall \lambda \in (0, \infty)$ $\exists$ $c = c(\lambda) \in (0,1)$ s.t.

$$\lim_{n \to \infty} \phi^{w}_{p_c(q),q} \Big[\quad \Big] = c(\lambda)$$



Conjecture: Fix $q \geq 1$. Then there exists $q_c(d) \in [2, \infty)$ s.t.

(i) $\phi^b_{p_c(q),q}(0 \longleftrightarrow \infty) = 0$ for every $q \geq 1$

(ii) $\phi^w_{p_c(q),q}(0 \longleftrightarrow \infty) \begin{cases} = 0 & \text{if } q \leq Q_c(d) \\ > 0 & \text{if } q > Q_c(d) \end{cases}$

(continuous phase transition)

(discontinuous phase transition)

It is believed that $Q_c(d) = 2$ if $d \geq 6$

Rk: • the conjecture is proved in dimension $d = 2$. (with $q_c(2) = 4$ (see previous Thm))

• It is known that

• $\phi^w_{p_c(q),q}(0 \longleftrightarrow \infty) = 0$ for $q = 2$ (Ising) for every $d \geq 2$

• $\phi^w_{p_c(q),q}(0 \longleftrightarrow \infty) > 0$ for fixed $d \geq 2$ and q large enough.

• $\phi^w_{p_c(q),q}(0 \longleftrightarrow \infty) > 0$ for $q \in \{3, \dots \infty\}$ and d large enough.

Let us mention a particular case, when $q = 1$:

Conjecture:

Consider Bernoulli percolation P_p on $\mathbb{Z}^d$ at $p = p_c(1)$.

Then $P_{p_c}[0 \longleftrightarrow \infty] = 0$

The conjecture has been proved for $d = 2$ and $d \geq 11$.

A major open problem is to solve the conjecture for $d = 3$..

The results and conjectures are presented on the following diagram (the results are in black, the conjectures in blue)

0 $\quad \phi^w_{p,q}[0\leftrightarrow\infty)=0 \quad$ $p_c(q)$ $\quad \phi^w_{p,q}[0\leftrightarrow\infty)>0 \quad$ 1

[EXPONENTIAL DECAY]

$G_n=[-n,n]^d \qquad \phi^w_{G_n,p,q}[0\leftrightarrow\partial G_n]\leq e^{-cn}$

[UNIQUENESS]

$\phi^w_{p,q}=\phi^b_{p,q}$

[MEAN-FIELD LOWER BOUND]

$\phi^w_{p,q}[0\leftrightarrow\infty)\geq c\,(p-p_c(q))$

[UNIQUENESS]

$\phi^w_{p,q}=\phi^b_{p,q}$

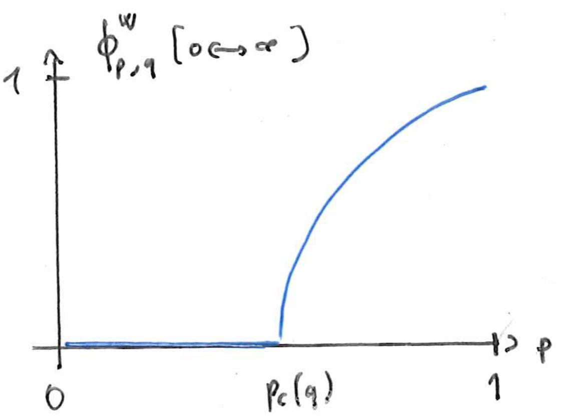
[UNIQUENESS OF THE INFINITE CLUSTER]

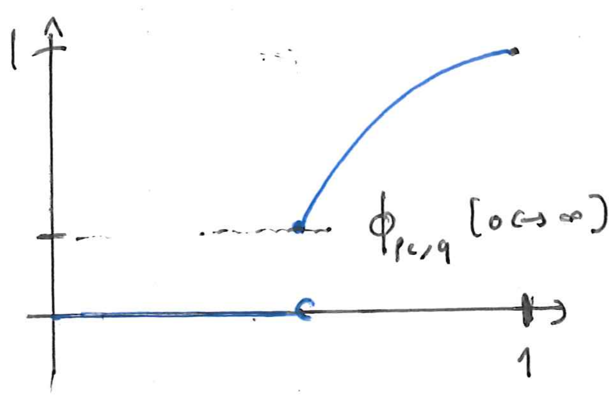
$\phi^w_{p,q}[\exists \text{ unique infinite cluster}]=1$

$$\boxed{\begin{array}{l}\phi^b_{p_c,q}[0\leftrightarrow\infty]=0\\ \phi^w_{p_c,q}[0\leftrightarrow\infty]\begin{cases}=0 & \text{if } q\leq Q_c(d)\\ >0 & \text{if } q> Q_c(d)\end{cases}\end{array}}$$

Expected behaviour of the percolation density

Continuous phase transition ($q\leq q_c(d)$) $\qquad$ discontinuous phase transition ($q>q_c(d)$)





5) Consequences for ISING MODEL. $(q=2)$

Thm: Let β_c be the critical inverse-temperature for Ising model on $\mathbb{Z}^d$.

$$p_c(2) = 1 - e^{-2\beta_c}.$$

For all $\beta < \beta_c$, we have.

(i) $\langle \sigma_0 \sigma_x \rangle_\beta^{\emptyset} \leq e^{-c|x|}$

(ii) $\langle \sigma_0 \rangle_{\Lambda_n,\beta}^{+} \leq e^{-cn}$ if $\Lambda_n = [-n,n)$

Proof: let $p = 1 - e^{-2\beta}$, then

$$\langle \sigma_0 \rangle_\beta^{+} = \lim_{n\to\infty} \langle \sigma_0 \rangle_{\Lambda_n,\beta}^{+}.$$

$$= \lim_{n\to\infty} \phi_{G_n,p,2}^{w}(0 \longleftrightarrow \partial G_n) \qquad (G_n = (\overline{\Lambda_n}, \overline{E_{\Lambda_n}}))$$

$$= \phi_{p,2}^{w}(0 \longleftrightarrow \infty)$$

This implies directly $p_c(2) = 1 - e^{-2\beta_c}$.

To prove (i), observe that

$$\langle \sigma_0 \sigma_x \rangle_\beta^{\emptyset} = \phi_{p,2}^{f}(0 \longleftrightarrow x)$$

$$\leq \phi_{p,2}^{f}(0 \longleftrightarrow \partial B_{|x|})$$

$$\leq \phi_{B_{|x|},p,2}^{w}(0 \longleftrightarrow \partial B_{|x|})$$

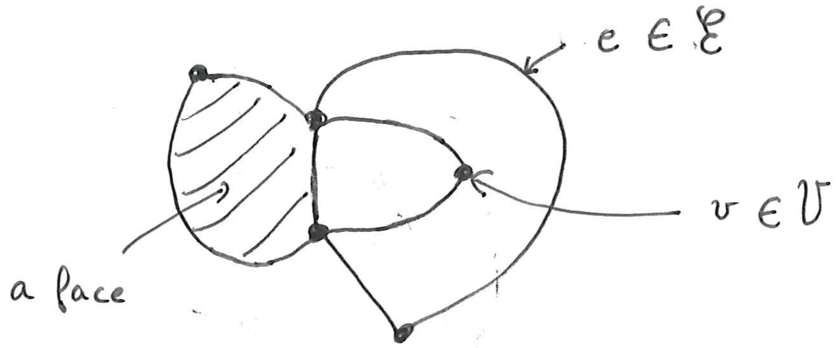
$$\leq e^{-c|x|}.$$

$\square$

PLANAR FK-PERCOLATION

1. FK-PERCOLATION ON A FINITE PLANE GRAPH.

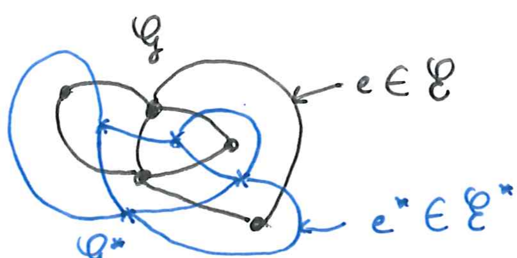
Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a finite connected plane graph. A vertex is a point in $\mathbb{R}^2$, an edge is an arc between two vertices. Any edge intersects no vertex and no other edge (except at its extremities).



A plane graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

A face of $\mathcal{G}$ is a connected component of $\mathbb{R}^2 \setminus \mathcal{G}$.

A plane graph $\mathcal{G}^*$ is a dual of $\mathcal{G}$ if each face of $\mathcal{G}$ contains exactly one vertex of $\mathcal{G}^*$, each vertex of $\mathcal{G}$ is contained in exactly one face of $\mathcal{G}^*$, and each edge e of $\mathcal{G}$ is crossed by exactly one edge e^* of $\mathcal{G}^*$ at one point.



A plane graph $\mathcal{G}$ and one of its dual (in blue)

As before, we can define FK percolation on $\mathcal{G}$ by setting for $p \in [0,1]$ and $q > 0$, for any $\omega \in \{0,1\}^{\mathcal{E}}$

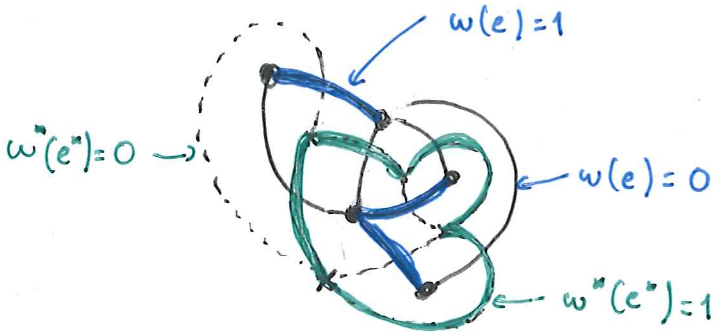
$$\boxed{\phi_{\mathcal{G},p,q}(\omega) = \frac{1}{Z_{\mathcal{G},p,q}} \, p^{o(\omega)} (1-p)^{c(\omega)} q^{k(\omega)}}$$

where $k(\omega)$ is the number of connected components of ω, identified to the subgraph $(V, \{e : \omega(e) = 1\})$.

Rk: note that here there is no boundary condition.

The dual configuration $\omega^* \in \{0,1\}^{\mathcal{E}^*}$ of ω is defined by

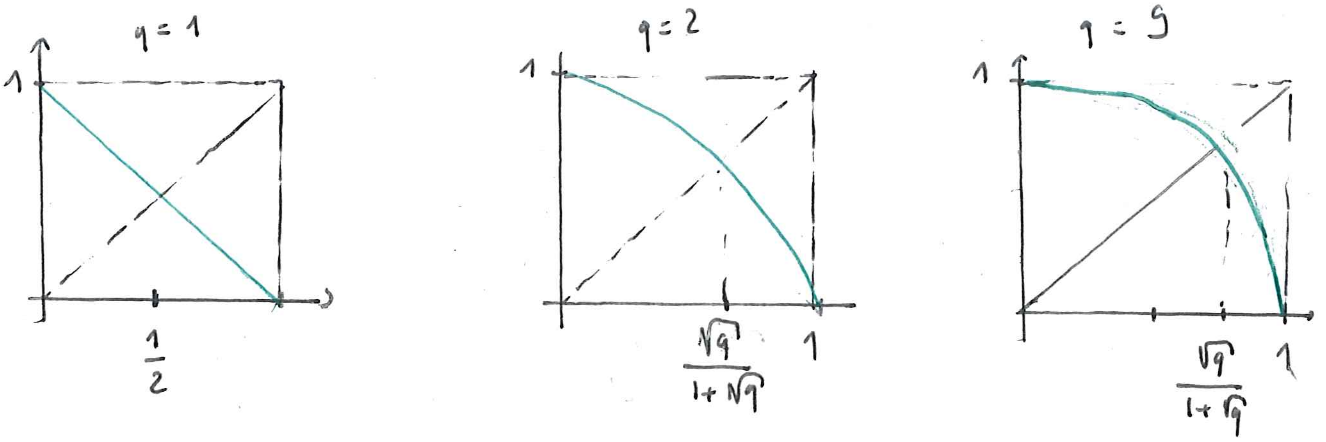
$$\forall e \in \mathcal{E} \quad \omega^*(e^*) = 1 - \omega(e)$$



Thm. For every $\omega \in \{0,1\}^{\mathcal{E}}$, we have

$$\phi_{\mathcal{G},p,q}(\omega) = \phi_{\mathcal{G}^*,p^*,q}(\omega^*) \qquad (1)$$

where $\dfrac{p^*}{1-p^*} = \dfrac{q(1-p)}{p}$



graph of $p \mapsto p^* = \dfrac{(1-p)\,q}{p + (1-p)q}$ for different q

Note that $p_{sd} := \dfrac{\sqrt{q}}{1+\sqrt{q}}$ is the unique fixed point of the mapping $p \longrightarrow p^*$.

<u>Proof</u> We write Cte for a constant independent of w (it may depend on $\mathcal{G}, p, q$, and change from line to line). Note that

$$\phi_{\mathcal{G},p,q}(w) = \text{Cte} \cdot \left(\frac{p}{1-p}\right)^{\sigma(w)} \cdot q^{k(w)}$$

In order to prove (1) we express $\sigma(w)$ and $k(w)$ in term of $\sigma(w^*)$ and $k(w^*)$. First, we have

$$\sigma(w) = |\mathcal{E}| - \sigma(w^*).$$

In order to express $k(w)$, we apply Euler formula to the graph $\mathcal{G}_{w^*} = (\mathcal{V}^*, \{e : w^*(e) = 1\})$. Writing $f(w^*)$ for the number of faces of $\mathcal{G}_{w^*}$, we have

$$\underset{|\text{vertices}|}{\underset{\uparrow}{|\mathcal{V}^*|}} - \underset{|\text{edges}|}{\underset{\uparrow}{\sigma(w^*)}} + \underset{|\text{faces}|}{\underset{\uparrow}{f(w^*)}} = 1 + \underset{|\text{connected components}|}{\underset{\uparrow}{k(w^*)}}.$$

We claim that

$$f(\omega^*) = k(\omega) \qquad (2)$$

This statement is rather intuitive since the clusters of ω are exactly included in the faces of ω^* (see the picture at the beginning of the chapter.). We defer the rigorous proof of (2) to the end of the proof. Plugging (2) in Euler formula we get

$$k(\omega) = 1 + k(\omega^*) + o(\omega^*) - |V^*|$$

<u>Hence</u>

$$\phi_{\mathcal{G},p,q}(\omega) = \frac{1}{Cte}\left(\frac{p}{1-p}\right)^{|\mathcal{E}| - o(\omega^*)} q^{1-|V^*| + k(\omega^*) + o(\omega^*)}$$

$$= \frac{1}{Cte}\left(\frac{q(1-p)}{p}\right)^{o(\omega^*)} q^{k(\omega^*)}$$

$$= \frac{1}{Cte}\left(\frac{p^*}{1-p^*}\right)^{o(\omega^*)} q^{k(\omega^*)}$$

$$= \phi_{\mathcal{G}^*,p^*,q}(\omega^*)$$

It remains to prove (2). Since the edges of $\mathcal{G}_\omega$ cannot intersect the edges of $\mathcal{G}_{\omega^*}$, the clusters of $\mathcal{G}_\omega$ must be included in the faces of $\mathcal{G}_{\omega^*}$, hence

$$f(\omega^*) \leq k(\omega)$$

Equivalently we have $\ell(\omega) \leq k(\omega^*)$

By Euler formula (applied to $\mathcal{G}_\omega$ and $\mathcal{G}_{\omega^*}$), this implies

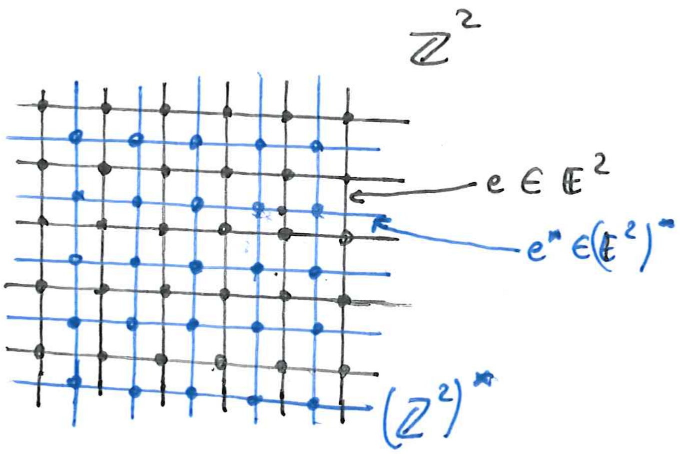
$$1 + k(\omega) + o(\omega) - |V| \leq |V^*| - o(\omega^*) + \ell(\omega^*) - 1$$

Since $|V| - (o(\omega) + o(\omega^*)) + |V^*| = 2$ (by Euler formula applied to the graph $\mathcal{G}$), we finally get

$$k(\omega) \leq \ell(\omega^*) \qquad \blacksquare$$

2. DUALITY ON THE SQUARE LATTICE

Let $((\mathbb{Z}^2)^*, (\mathbb{E}^2)^*)$ be the graph obtained by translating $(\mathbb{Z}^2, \mathbb{E}^2)$ by vector $(\frac{1}{2}, \frac{1}{2})$



each edge of $\mathbb{E}^2$ crosses a unique edge of $(\mathbb{E}^2)^*$, denoted e^*.

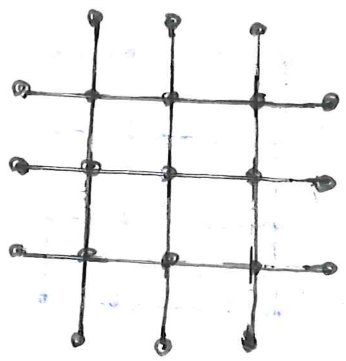
Given $\omega \in \{0,1\}^{\mathbb{E}^2}$ we define $\omega^* \in \{0,1\}^{(\mathbb{E}^2)^*}$ by setting

$$\omega^*(e^*) = 1 - \omega(e)$$

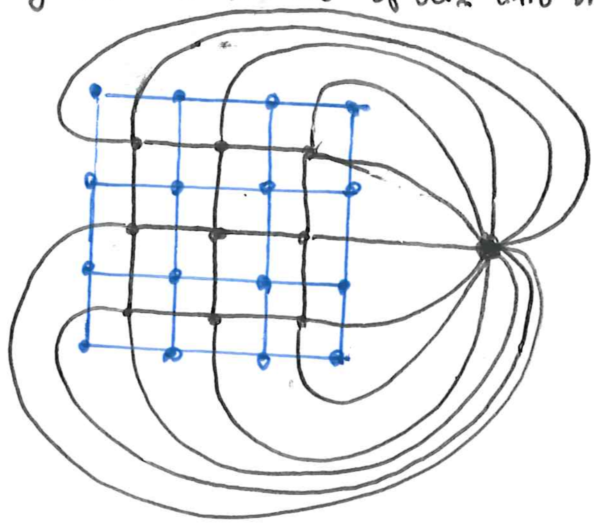
⑥

Thm: Let X be a FK random configuration with law $\phi^{w}_{p,q}$. Then X^* has law $\phi^{f}_{p^*,q}$ on $(\mathbb{Z}^2)^*$

Proof: Let $B_m = \{-m, \dots, m\}$ and G_m the subgraph of $\mathbb{Z}^2$ induced by $E_{\overline{B_m}}$. Let $\mathcal{G}_m$ be the plane graph obtained from G_m by contracting all the vertices of ∂G_m onto one vertex



G_m



The plane graph $\mathcal{G}_m$ and its dual $\mathcal{G}_m^*$ (in blue)

Notice that the subgraph G_m^* of $(\mathbb{Z}^2)^*$ induced by $\{-m+\frac{1}{2}, \dots, m-\frac{1}{2}\}^2$ is a dual of $\mathcal{G}_m$. Furthermore a configuration ω on G_m can be seen as a configuration on $\mathcal{G}_m$, and

$$\phi^{w}_{G_m,p,q}(\omega) = \phi_{\mathcal{G}_m,p,q}(\omega) = \phi^{f}_{G_m^*,p^*,q}(\omega^*)$$

Hence, for every local event A

$$\phi^{w}_{p,q}(A) = \phi^{f}_{p^*,q}(A^*)$$

where $A^* = \{ \omega^*, \omega \in A \}$.

Hence for every local event.

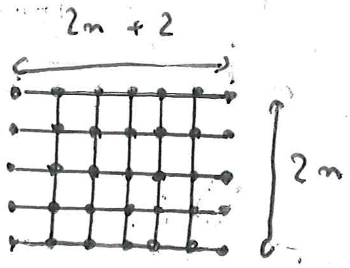
$$\mathbb{P}[X^* \in A^*] \overset{\text{def of } X^*}{=} \mathbb{P}[X \in A]$$
$$= \phi^{w}_{p,q}(A)$$
$$= \phi^{f}_{p^*,q}(A^*)$$

<u>Remark:</u> In general it is possible to define a notion of a planar b.c. ξ and their dual ξ^*, in such a way that for every ω configuration on G_n

$$\phi^{\xi}_{G_n,p,q}(\omega) = \phi^{\xi^*}_{G_n^*,p^*,q}(\omega^*)$$

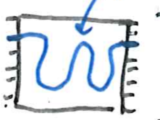
In the proof above we proved it for $\xi = w$ and $\xi^* = f$.

<u>Exercise:</u> Let

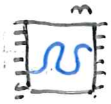
$G_n =$ 

Consider the b.c. mix: wired on the left and right, and free on top and bottom. Prove that

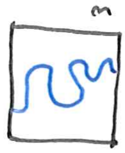
path from left to right

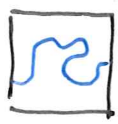
$$\phi^{mix}_{G_n,p_{sd},q}\Big[$$  $$\Big] = \frac{\sqrt{q}}{1+\sqrt{q}}$$

3) Computation of $p_c(q)$

Let G_n be the graph illustrated in the exercice above.
consider the event  that there exists a path from left to right in G_n in the configuration ω.

Lemma: Fix $q \geq 1$

1) For $p < p_c(q)$, we have $\phi^w_{p,q}$ [] $\xrightarrow[n\to\infty]{} 0$

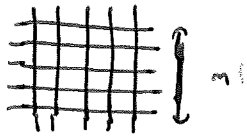
2) For $p > p_c(q)$, we have $\phi^f_{p,q}$ [] $\xrightarrow[n\to\infty]{} 1$

Proof:

1) Let $L \subset \partial G_n$ be the left-side of G_n. If there is a path from left to right in G_n, then there must exist a point $x \in L$ that is connected to distance n around it. Therefore, by translation invariance,

$\phi^w_{p,q}$ () $\leq |L| \times \phi^w_{p,q}(0 \longleftrightarrow \partial B_n)$

$$\leq (2n+1)\, e^{-cn} \xrightarrow[n\to\infty]{} 0 .$$

2) Consider the graph $\Lambda_m =$ 

we partition $\partial\Lambda_n = \underset{\text{left}}{\underset{\uparrow}{S_1}} \cup \underset{\text{right}}{\underset{\uparrow}{S_2}} \cup \underset{\text{top}}{\underset{\uparrow}{S_3}} \cup \underset{\text{bottom}}{\underset{\uparrow}{S_4}}$.

Let $\varepsilon > 0$. Pick m large enough such that

$$\phi^b_{p,q}[\Lambda_m \longleftrightarrow \infty] \geq 1 - \varepsilon^4.$$

(possible because $\phi^b_{p,q}(0 \longleftrightarrow \infty) > 0$)

In particular, for every $n \geq m$

$$\phi^b_{p,q}[\Lambda_m \longleftrightarrow \partial\Lambda_n] \geq 1 - \varepsilon^4$$

ie

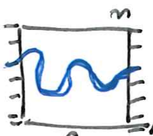
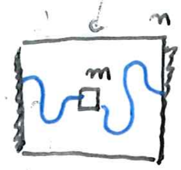
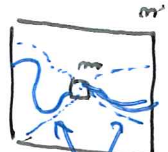
$$\phi^b_{p,q}\Big[\bigcup_{i=1}^{4} (\Lambda_m \longleftrightarrow S_i)\Big] \geq 1 - \varepsilon^4.$$

This implies for every i,

$$\phi^b_{p,q}[\Lambda_m \longleftrightarrow S_i] \geq 1 - \varepsilon$$

(indeed, by FKG inequality and symmetry, we have

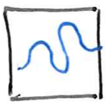
$$\phi^b_{p,q}[\Lambda_m \not\longleftrightarrow S_1]^4 \underset{\text{FKG}}{\leq} \phi^b_{p,q}[\Lambda_m \not\longleftrightarrow \partial\Lambda_n] \leq \varepsilon^4 \;)$$

Now $\phi^{f}_{p,q}\Big($  $\Big) \geq \underbrace{\phi^{b}_{p,q}\Big(}$  $\underbrace{\Big)}_{\geq (1-\varepsilon)^2} - \phi^{f}_{p,q}\Big($  $\underbrace{\Big)}_{\substack{\to 0 \\ n\to\infty}}$

G_n

disjoint clusters

(by uniqueness)

Hence, $\liminf_{n\to\infty} \phi^{b}_{p,q}\Big($  $\Big) \geq (1-\varepsilon)^2$ ∎

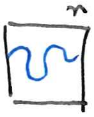
Thm: For FK-percolation on $\mathbb{Z}^2$, we have

$$\boxed{p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}}$$

Proof: $p_{sd} = \dfrac{\sqrt{q}}{1+\sqrt{q}}$ is such that $p_{sd}^{*} = p_{sd}$

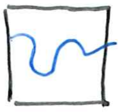
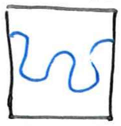
Let $X \sim \phi^{w}_{p,q}$ (Hence $X^{*} \sim \phi^{f}_{p,q}$)

We have

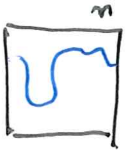
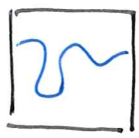
$\underbrace{\mathbb{P}\big[}$  $\underbrace{\text{in } X\big]}_{= \phi^{w}_{p_{sd},q}(\text{box } n)}$ $+$ $\underbrace{\mathbb{P}\big[}$  $\underbrace{\text{in } X^{*}\big]}_{= \phi^{f}_{p_{sd},q}(\text{box } n)}$ $= 1$

$= \phi^{w}_{p_{sd},q}\Big($  $\Big)$

$\underset{\substack{\uparrow \\ \text{sym} \\ + p_{sd}^{*} = p_{sd}}}{=} \phi^{f}_{p_{sd},q}\Big($  $\Big)$

Since $\phi^{w}_{p_{sd},q}\Big($  $^{n}\Big) \geq \phi^{b}_{p_{sd},q}\Big($  $^{n}\Big)$

we obtain, for every n

$\phi^{w}_{p_{sd},q}\Big($  $^{n}\Big) \geq \frac{1}{2}$ and $\phi^{b}_{p_{sd},q}\Big($  $^{n}\Big) \leq \frac{1}{2}$

and therefore, by the Lemma $\boxed{p_{sd} = p_c}$ ■

<u>Corollary</u>:

For Ising model on the square lattice, we have

$$\beta_c = \frac{1}{2}\log\left(1+\sqrt{2}\right)$$