FK-percolation (also called "the random-cluster model") is a percolation process, invented in 1969 by Cees Fortuin and Piet Kasteleyn. We would like to give two main motivations for this 1-parameter family of processes (indexed by \( q \geq 1 \)):

1) it is a generalization of Bernoulli bond percolation (which corresponds to the case when \( q = 1 \)), and it shares many properties with Bernoulli percolation. In particular, several techniques developed for Bernoulli percolation will also apply for FK-percolation. This makes the process very suitable for a mathematical and rigorous analysis.

2) it is related to other models in statistical mechanics. For \( q = 2 \), the model is related to Ising model; first we will see that it can be seen as a geometric representation of Ising model by expanding the partition function, but in the course we will establish a deeper relation based on the Edwards-Sokal coupling. More generally, for \( q \geq 2 \) integer, FK percolation is related to Potts model, a generalization of Ising to more than two possible spin-values.
In this second part of the course, we will:

- define FK-percolation on a finite subgraph of \( \mathbb{Z}^d \) and present basic properties of the model. As for Ising model, we will see that boundary conditions play an important role.

- present the connections with Ising model via the Edwards-Sokal coupling.

- define FK percolation in infinite-volume and define the phase transition. This part will be very similar to the corresponding one for the Ising model.

- prove a fundamental result concerning sharpness of the phase transition.

- discuss the planar case (on \( \mathbb{Z}^2 \)). Based on duality we will be able to compute the critical value.
Before defining the model, we describe a new geometric representation for the Ising model with free b.c. and no external magnetic field.

Recall that the Hamiltonian of the free Ising model on $\Lambda \subset \mathbb{Z}^d$ is

$$H_{\Lambda, \beta, 0}(\sigma) = -\beta \sum_{i, j \in E_\Lambda} \sigma_i \sigma_j.$$

$$= -\beta |E| + 2\beta \sum_{i, j \in E_\Lambda} \text{if } \sigma_i \neq \sigma_j.$$

Therefore

$$Z_{\Lambda, \beta, 0} = \sum_{\sigma \in \Omega_\Lambda} e^{-H_{\Lambda, \beta, 0}(\sigma)}.$$

$$= \sum\limits_{\sigma \in \Omega_\Lambda} e^{\frac{\beta |E|}{2}} \prod_{i,j \in E_\Lambda} \left( e^{-2\beta} + (1-e^{-2\beta}) \text{if } \sigma_i = \sigma_j \right).$$

$$= e^{\frac{\beta |E|}{2}} \sum_{\sigma \in \Omega_\Lambda} e^{-2\beta c(\omega)} (1-e^{-2\beta})^{\sum_{i,j \in \Omega_\Lambda} \text{if } \sigma_i = \sigma_j \text{and } \omega(i,j) = 1}.$$

where

$$c(\omega) = \left| \{i,j \in E_\Lambda : \omega(i,j) = 1 \} \right| \text{ "open edges on } \omega\text{"}$$

$$= \left| \{i,j \in E_\Lambda : \omega(i,j) = 0 \} \text{ "closed } \omega\text{"} \right|$$

$$= \sum_{\omega \in \{0,1\}^E_\Lambda} e^{-2\beta c(\omega)} (1-e^{-2\beta})^{\sum_{i,j \in \Omega_\Lambda} \text{if } \sigma_i = \sigma_j \text{and } \omega(i,j) = 1} \text{ if } \sigma \text{ is constant on each connected component of } \omega.$$
For fixed $w \in \{0,1\}^{E_n}$, the number of spin configurations $\sigma \in \mathbb{R}^n$ that are constant in the connected components of $w$ is exactly $2^{k(w)}$, where $k(w)$ is the number of connected components in $w$. (There is one possible value, + or -, for each connected component.)

Finally, we get the following expansion of the partition function:

$$Z^{\phi, \beta, \sigma} = e^{+\beta |E|} \sum_{w \in \{0,1\}^{E_n}} p^{\sigma(w)} (1-p)^{\bar{\sigma}(w)} 2^{k(w)}$$

where $p = 1 - e^{-2\beta}$
CHAPTER 1: FK-PERCOLATION.
ON A FINITE GRAPH.

DEFINITIONS

- We call subgraph of $\mathbb{Z}^d$ a pair $G = (V,E)$ where $E \subset \mathbb{Z}^d$ is the set of edges and $V = U_{(e)}$ is the set of vertices induced by $E$.

- A subgraph of $\mathbb{Z}^2$ with 11 edges and 10 vertices. The vertex boundary is represented by the red vertices.

- The vertex boundary of $G$ is defined by $\partial G := \{ x \in V : \exists y \in \mathbb{Z}^d \setminus V, \{x,y\} \in E \}$.

- A boundary condition (b.c.) is a partition of $\partial G$.

- A percolation configuration is an element $w = (w(e), \in E) \in \{0,1\}^E$.

- One can identify a percolation configuration $w$ to the subgraph of $G$ given by $(V', \{e \in E : w(e)=1\})$.

A graph $G$

A percolation configuration in $G$.
We denote by
\[ \sigma(\omega) = \| e \in E : \omega(e) = 1 \| \] the number of open edges.
\[ \phi(\omega) = \| e \in E : \omega(e) = 0 \| \] closed

Given a b.c. \( \Psi = \{ p_1, \ldots, p_k \} \) and a configuration \( \omega \in \{0,1\}^E \), we define \( k_\Psi(\omega) \) to be the number of clusters in \( \omega \) where all the clusters intersecting the same \( p_i \) count as one.

**Example:**

\[ G = \] 

\[ \omega = \] 

\[ \sigma(\omega) = 5 \] 

\[ \phi(\omega) = 5 \] 

\[ k_\Psi(\omega) = 5 \] 

\[ k_\Psi(\omega) = 2 \] 

The element of \( \Psi \) are "wired" together.

**Definition:**

Let \( p \in [0,1] \), \( q > 0 \). The \( F_k \)-pencolation measure on \( G \) with edge-weight \( p \), cluster-weight \( q \) and b.c. \( \Psi \), \( \omega \) defined by

\[ \forall \omega \in \{0,1\}^E \quad \phi_{G, p, q} (\omega) = \frac{p^{\sigma(\omega)} (1-p)^{\phi(\omega)} q^{k_\Psi(\omega)}}{Z_{G, p, q}} \]

where \( Z_{G, p, q} = \sum_{\omega \in \{0,1\}^E} p^{\sigma(\omega)} (1-p)^{\phi(\omega)} q^{k_\Psi(\omega)} \)
Rks:

1. For $q = 1$ we have $\Phi_{\lambda, r, q}(w) = L \phi_0(\lambda)(1-p)^{\lambda}$ and therefore $\Phi_{\lambda, r, 1}$ coincides with Bernoulli bond percolation.

2. As $q$ increases, the measure "favors" configurations with more disjoint clusters.

3. Even if we will not consider these objects in this class, we want to mention that the limit as $q$ tends to 0 is related to other important objects in statistical mechanics.

$$\Phi_{r, q} \underset{q \to 0}{\longrightarrow} \begin{cases} 
\text{"uniform connected subgraph" if } p = \frac{1}{2}, \\
\text{"uniform spanning tree" if } p \neq 0 \text{ and } q \to 0, \\
\text{"uniform spanning forest" if } p = q.
\end{cases}$$

2) **Main Properties**

**Lemma:**

Let $e \in E$, $w \in \{0, 1\}^E$. Then

$$\Phi_{\lambda, r, q}(we) = \begin{cases} 
\frac{p}{1-p} & \text{if the extremities of } e \text{ are connected in } (we, q), \\
\frac{p^q}{q(1-p)} & \text{otherwise}.
\end{cases}$$

We say that two vertices are connected in $(w, q)$ if there are connected in $w$ or they are connected to the same element of the partition $\mathcal{Y}$.
Recall: Let \( E \subseteq \{x,y\} \in E \)

\[
\phi_{\psi_{x,y}}^p \left[ \omega(e) = 1 \mid \forall b \in \omega(b) = \psi(b) \right] = \frac{1}{1 + \phi[\psi_x]/\phi[\psi_y]}
\]

\[
= \begin{cases} 
1 & \text{if } x \to y \text{ in } (\omega, \psi) \\
\frac{pq}{pq + (1-p)} & \text{otherwise}
\end{cases}
\]

**Theorem (Domain Markov Property)**

Let \( G' = (V', E') \) be a subgraph of \( Z^d \) of \( E \). Let \( p \in (0,1) \), \( q > 0 \), \( p \) b.c. \( p \geq G \).

Then for all \( \psi \in \{0,1\}^E \), \( \forall \gamma \in \{0,1\}^E \),

\[
\phi_{\gamma, p, q} \left[ \omega_{i,e} = \gamma \mid \omega_{i,e} = \psi \right] = \phi_{\psi, p, q}^\gamma
\]

where \( \phi^\gamma \) is the partition of \( Z^d \) induced by \( \gamma \) and \( \phi^\gamma : x, y \in Z^d \) are in the same element of the partition if \( x \) is connected to \( y \) in \((\psi, \gamma)\).
From now on and until the end of the course, we always assume

\[ q \geq 1 \]

**Theorem [FKG Inequality]**

Let \( p \in (0,1) \), \( q \geq 1 \). Let \( p \) be constant.

For every \( A, B \subset \{0,1\}^E \) increasing events,

\[
\phi^{p}_{A, p, q}(A \cap B) \geq \phi^{p}_{A, p, q}(A) \phi^{p}_{A, p, q}(B) \quad \text{[FKG]}
\]

**Proof:**

By the lemma, we have for every \( \gamma \leq \psi \) and every \( e \in (\gamma, \psi) \)

\[
\phi^{p}_{A, p, q}(\gamma \cap e) = \frac{p}{1-p} \phi^{p}_{A, p, q}(\gamma) \leq \frac{p}{1-p} \phi^{p}_{A, p, q}(\psi \cap e) \]

\[
\phi^{p}_{A, p, q}(\gamma \cap e) \leq \frac{p}{1-p} \phi^{p}_{A, p, q}(\psi \cap e) \]

\[
\phi^{p}_{A, p, q}(\gamma \cap e) \leq \frac{p}{1-p} \phi^{p}_{A, p, q}(\psi \cap e)
\]

Holley criterion for FKG inequality concludes the proof.

**Theorem [Comparison Between B.C.]**

Let \( p \in (0,1) \), \( q \geq 1 \).

Let \( \psi \leq \psi' \) two b.c. (meaning that any element of the partition \( \psi \) is included in an element of \( \psi' \)). Then

\[
\phi^{p}_{A, p, q}(\psi) \ll \phi^{p}_{A, p, q}(\psi')
\]
In particular, we have for every b.c. \( \Psi \),

\[
\phi_{\alpha, r, q} [\Psi] \ll \phi_{\alpha, r', q} [\Psi] \ll \phi_{\alpha, r', q'} [\Psi].
\]

**Proof:** As above, if \( \gamma \leq \Psi \), we can use that

\[
(\Psi \leq 1) \iff (\gamma \leq 1) \iff (\gamma \leq \Psi)
\]

to show

\[
\frac{\phi_{\alpha, r, q} [\gamma]}{\phi_{\alpha, r, q} [\gamma_c]} \ll \frac{\phi_{\alpha, r, q} [\Psi]}{\phi_{\alpha, r, q} [\Psi_c]},
\]

and then apply the Holley argument.

**Theorem:** (Monotonicity Properties)

Let \( \Psi \) be a b.c.

- If \( p \leq p' \) and \( q \geq q' \), then \( \phi_{\alpha, p, q} [\Psi] \ll \phi_{\alpha, p', q'} [\Psi] \) \((p, q) \rightarrow \phi_{\alpha, p, q} \) is increasing in \( p \), decreasing in \( q \).

- If \( 1 \leq p \leq q \) and \( \frac{1}{q(1-p)} \leq \frac{1}{q'(1-p')} \), then \( \phi_{\alpha, p, q} [\Psi] \ll \phi_{\alpha, p', q'} [\Psi] \) \( (\text{even if } q' = q', \text{ taking } p' \text{ sufficiently large then } p' \text{ gives rise to the domination } \phi_{p, q} [\Psi] \ll \phi_{p', q'} [\Psi] ) \).
Proof: To prove the two results it suffices to prove that for every \( p \leq p' \) and \( q \leq q' \),

\[
\frac{p}{q(1-p)} \leq \frac{p'}{q'(1-p')} \Rightarrow \phi_{p, q} \leq \phi_{p', q'}
\]

Again, we use the Holley criterion. The inequalities \( \frac{p}{q(1-p)} \leq \frac{p'}{q'(1-p') \quad \text{and} \quad p \leq p' \) imply for all \( a, b \in [0, 1] \) and \( a \leq b \),

\[
\frac{p}{q(1-p)} q^a \leq \frac{p'}{q'(1-p')} q'^b
\]

(distinguish between the 3 possible cases)

Therefore \( \forall q \leq q' \),

\[
\frac{p}{q(1-p)} \underbrace{\mu_{x=y} \in [q', q]}_{a} \leq \frac{p'}{(1-p)q'} \underbrace{\mu_{x=y} \in [q, q']}_{b}
\]

Applications \( \forall \in \mathbb{E} \)

1. \( \phi_{p, p, q} \quad \text{if} \quad \omega(x) = 1 \) in \( p \) and decreasing in \( q \).

2. \( \phi_{p, p', q} \ll \phi_{p, p, q} \ll \phi_{p, p, 1} \quad \text{for} \quad p' = \frac{p}{p + q(1-p)} \)

Bouma (p)
Given $\Lambda \subseteq \mathbb{Z}^d$, we consider the graph $G = (\Lambda, E_\Lambda)$.

Let $p \in [0, 1]$, let $w \in \{0, 1\}^{E_\Lambda}$.

To each cluster $C$ of $w$, associate a random variable $Z_C$ such that $(Z_C)$ are independent.

$\Pr(Z_C = +1) = \frac{1}{2}$

Define $Y(w)$ by setting for every cluster $C$ and every edge $e$:

$Y(e) = Z_C$

We say that $Y$ is a coloring of the cluster of $p$.

Then: [Coupling between $p_0$ and $p_\infty$]

Let $p \in [0, 1]$ and $\beta > 0$ such that $p = 1 - e^{-2\beta}$.

If $X \sim p_\infty$, then an independent coloring $Y$ of $w^*$ is on has the law $p_\infty$. 
Let \( \sigma \in \Omega \Lambda \). Let \( D_\sigma = \{ \lambda, \gamma \in E : \sigma_\lambda \neq \sigma_\gamma \} \).

Define \( X' \in \{0,1\}^E \) by
\[
    P[X(e) = 1] = \begin{cases} 
        1 & \text{if } e \notin D_\sigma \\
        0 & \text{if } e \in D_\sigma
    \end{cases}
\]

in such a way that \( (X(e)) \) are independent.

We say that \( X' \) is a \( p \)-percolation on \( E \setminus D_\sigma \).

Then:
\[
    \text{Let } p > 0 \text{ and } p = 1 - e^{-2\beta}
\]

1. If \( X \sim \phi_{\alpha, \rho, 2} \), and \( Y \) is an ind. coding of \( X \),
   then \( Y \sim \psi_{\alpha, \beta, 0} \).

2. If \( Y' \sim \psi_{\alpha, \beta, 0} \), and \( X' \) is a \( p \)-percolation in \( E \setminus D_\sigma \),
   then \( X' \sim \phi_{\alpha, \rho, 2} \).

Remark: We will actually prove that \( (X, Y) \) and \( (X', Y) \)
have the same law. This common law is a coupling
of \( \phi_{\alpha, \rho, 2} \) and \( \psi_{\alpha, \beta} \).
We say $\sigma$ is compatible with $w$ if for every $x, y \in \mathcal{E}$,

$$(x \rightarrow y \in w) \Rightarrow (\sigma x = \sigma y)$$

In other words, $\sigma$ is compatible with $w$ if it is constant on its connected components.

$$\mathbb{P}[X = w, Y = \sigma] = \frac{1}{Z_{G, p, 2}} \sum_{x \in w} \sigma(x)^{\ell_G(x)} (1-p)^{c(x)}$$

By summing the two equations above over $\sigma, w$ we get

$$\mathbb{P}[X = w, Y = \sigma] = \frac{\mathbb{E}^*}{Z_{G, p, 2}} \sum_{x \in w} \sigma(x)^{\ell_G(x)} (1-p)^{c(x)}$$

we acquire the result

$$\frac{1}{Z_{G, p, 2}} = \frac{\mathbb{E}^*}{Z_{G, p, 2}}$$
Therefore, we obtain that for every $(ω,s) ∈ \{0,1\}^E × S^n$

$$P\left[ X = ω, Y = s \right] = P\left[ X' = ω', Y' = s' \right]$$

In particular, $Y$ has the same law as $Y'$, i.e. $Y \sim \mu^\phi_{\Lambda, β}$.

$X'$ has the same law as $X$, i.e. $X' \sim \phi_{a, p, 2}^b$.

**Rk:** We have proved that $(x, y)$ and $(x', y')$ have the same law $\mathbb{P}$ on $(\{0,1\}^E × S^n)$ given by $P[(ω,s)] = \frac{1}{2} \cdot \#\{ω\}$.

**Proposition:** $P$ is a coupling of $\phi_{a, p}$ and $\mu^\phi_{\Lambda, β}$.

Let $b > 0$ and $p = 1 - e^{-2b}$. Then, in every $i \in E$

We have

$$\langle σ_i σ_j \rangle_{\Lambda, β}^\phi = \phi_{a, p, 2}^b [ i \rightarrow j \text{ in } \omega ]$$

**Rk:** This implies directly that

$$\langle σ_i σ_j \rangle_{\Lambda, β}^\phi = \phi_{a, p, 2}^b [ i \rightarrow j \text{ in } \omega ]$$

Since the event $i \rightarrow j$ is increasing.

**Proof:**

Let $X \sim \phi_{a, p, 2}^b$, let $Y$ be a random coloring of the clusters of $X$. In particular $Y \sim \mu^\phi_{\Lambda, β}$ and we have

$$\langle σ_i σ_j \rangle_{\Lambda, β}^\phi = E \left[ Y_i Y_j \right]$$

$$= E \left[ Y_i Y_j \mid i \rightarrow j \text{ in } X \right] P \left[ i \rightarrow j \text{ in } X \right]$$

$$+ E \left[ Y_i Y_j \mid \neg i \rightarrow j \text{ in } X \right] P \left[ \neg i \rightarrow j \text{ in } X \right]$$

$$= 0$$

$$= \left[ i \rightarrow j \text{ in } \omega \right] \phi_{a, p, 2}^b \left[ i \rightarrow j \text{ in } \omega \right]$$
1) Coupling \( \phi^\infty \) and Ising.

Let \( \Lambda \subset \mathbb{Z}^d \). We consider the graph \( \widetilde{G}_\Lambda = (\widetilde{V}, \widetilde{E}) = (V, E_\Lambda) \) induced by the edges of \( E_\Lambda \) (\( V \) contains \( \Lambda \) and the vertices at the external boundary of \( \Lambda \)).

\[ \text{If } \Lambda = \Lambda_1 = \ldots \text{ Then } \widetilde{G} = \ldots \]

Rk: an Ising configuration \( \sigma \in \Omega_\Lambda^+ \) can be identified to an element \( \sigma \in \{-1, 1\}^V \) with \( \sigma = 0 \) on \( \partial G \).

Thm: Let \( p > 0 \) and \( p = 1 - e^{-2b} \).

1) Let \( X \sim \phi^\infty \) / let \( Y \) be a random coloring of the clusters of \( X \), with the additional condition that all the clusters touching \( \partial G \) are colored +1 (neop. -1) then \( Y \sim \phi^+_{\Lambda, p} \) (neop. \( Y \sim \phi^-_{\Lambda, p} \)).

2) Let \( Y' \sim \phi^+_{\Lambda, p} \), let \( X' \) be a \( p \)-percolation in \( E \setminus \partial Y \). Then \( X' \sim \phi^w_{G, p, 2} \)

Proof: Let \( \omega, \sigma \in \{0, 1\}^E \), \( \sigma \in \{-1, 1\}^V \) where \( \sigma(i) = +1 \) if \( i \in \partial \Lambda \).

\[
P[X = \omega, Y = \sigma] = \frac{1}{Z_{G, p, 2}} \prod_{w} p^\sigma(w) (1-p)^{1-\sigma(w)} 2^{k\omega(w)} \times \frac{1}{2^{k\omega(w)-1}}
\]

\[
= \frac{2}{Z_{G, p, 2}} \prod_{w} p^\sigma(w) (1-p)^{1-\sigma(w)}
\]
\[ \mathbb{P}[X = \omega, Y = \sigma] = \frac{1 + \beta |\bar{E}|}{2^{\lambda + \beta}} e^{-2\beta(1 - \rho)} \times \prod_{\sigma \in \omega} p(1 - p)^{c(\omega)} = \frac{\beta |\bar{E}|}{\lambda + \beta} \times \prod_{\sigma \in \omega} \frac{p^{\sigma(\omega)}(1 - p)^{c(\omega)}}{1 - \rho}. \]

**Proposition:**

Let \( \rho > 0 \) and \( p = 1 - e^{-2\beta} \). Then, if \( \sigma \in \Lambda \)

\[ <\sigma_0>^+_{\Lambda, \beta} = \phi_{\frac{\omega}{\sigma}}^{-1}(0 \in \partial \bar{G} \text{ in } \omega) \]

**Proof:**

Let \( X, Y \) be two random variables as in the above.

\[ <\sigma_0>^+_{\Lambda, \beta} = \mathbb{E}[Y_0] \]

\[ = \mathbb{E}[Y_0 \mid 0 \in \partial \bar{G} \text{ in } X] \mathbb{P}(0 \in \partial \bar{G} \text{ in } X) + \mathbb{E}[Y_0 \mid 0 \not\in \partial \bar{G} \text{ in } X] \mathbb{P}(0 \not\in \partial \bar{G} \text{ in } X) \]

\[ = \mathbb{P}(0 \in \partial \bar{G} \text{ in } X) \]

\[ = \phi_{\frac{\omega}{\sigma}}^{-1}(0 \in \partial \bar{G} \text{ in } \omega). \]
Proof of the GKS inequalities for \( h = 0 \)

**Theorem:** Let \( # \in \{+, \emptyset\} \) then \( \forall A, B \subset \mathbb{Z}^d \), we have

1. \( <\sigma_A>^#_{\land, \lor} > 0 \)
2. \( <\sigma_A \sigma_B>^#_{\land, \lor} > <\sigma_A>^#_{\land, \lor} <\sigma_B>^#_{\land, \lor} \)

We prove here these inequalities for \( # = \emptyset \).

**Lemma:** Let \( A, C \subset \mathbb{Z}^d \) and \( \mathbf{G} = (V, E) = (\mathbb{Z}^d, E_n) \).

Consider the percolation event

\[
\mathcal{F}_A = \{ w \in \{0,1\}^E : \not\exists C \in \mathcal{C}(w) : \left| C \cap A \right| \text{ is even} \}
\]

where \( \mathcal{C}(w) = \{ \text{clusters of } w \} \).

Then

\[
<\sigma_A>^+_{\land, \lor} = \phi_{\mathbb{Z}^d, p}(\mathcal{F}_A) \quad (p = 1 - e^{-2\theta})
\]

**Proof:**

Let \( X \sim \phi_{\mathbb{Z}^d, p} \) and \( \theta, \gamma \) a coloring of the cluster of \( X \).

\[
\mathbb{E}[<\sigma_A>^\#_{\land, \lor}] = \mathbb{E}\left[ \sum_{w \in \{0,1\}^E} \mathbb{E}[Y_A | X = w] \cdot \mathbb{P}[X = w] \right]
\]
Assume \( X = \omega \)

\[
Y_A = \prod_{c \in B(\omega) \cap A} Z_c = \prod_{c \in B(\omega)} Z_c^{1_{c \cap A}}
\]

\[
E(Y_A \mid X = \omega) = E\left( \prod_{c \in B(\omega)} Z_c \right) = \prod_{c \in B(\omega)} E[Z_c^{1_{c \cap A}}]
\]

= 1 \quad \text{if } \forall c \in B(\omega) \cap A \text{ is odd}

= 0 \quad \text{otherwise}

Therefore \( \langle \sigma_A \rangle_{\lambda, \beta} = \sum_{\omega \in \{0, 1\}} \prod_{\omega \in \sigma_A} P(X = \omega) \)

= \prod_{\omega \in \sigma_A} P(X = \omega)

= \Phi_{\alpha, \beta}^{b} \left[ \sigma_A \right]. \quad \Box

Proof of Theorem 2a: \( \# = \emptyset, A, B \) disjoint

1) \( \langle \sigma \cap \beta \rangle^0 \lambda, \beta = \Phi_{\alpha, \beta}^{b} \left[ \sigma_A \right] > 0 \)
\[ \langle \sigma_A \sigma_B \rangle \sigma = \langle \sigma_{A \cup B} \rangle \sigma \]

\[ = \mathcal{A} \phi \left( \mathcal{F}_{A \cup B} \right) \]

\[ \geq \phi \left( \mathcal{F}_A \cap \mathcal{F}_B \right) \quad \text{(because } \mathcal{F}_A \cap \mathcal{F}_B \subset \mathcal{F}_{A \cup B} \text{)} \]

\[ \geq \phi \left( \mathcal{F}_A \right) \phi \left( \mathcal{F}_B \right) \quad \text{(because } \mathcal{F}_A \mathcal{F}_B \uparrow \text{)} \]

\[ = \langle \sigma_A \sigma_B \rangle \sigma \]

\[ = \langle \sigma_A \rangle \langle \sigma_B \rangle \sigma. \]
9.1 $p \in (0,1)$ fixed.

1) THE WIRED AND FREE INFINITE-VOLUME MEASURES.

- $\Omega = \{0,1\}^d$ (edges of $\mathbb{Z}^d$)
- For $\mathcal{E} \subseteq \mathbb{Z}^d$, $\mathcal{F}_\mathcal{E} = \sigma$-algebra generated by $(\omega(e))_{e \in \mathcal{E}}$.

(Rk: if $A \in \mathcal{F}_\mathcal{E}$ and $\mathcal{G}(V,E)$ is the graph generated by $E$, then $\phi_{\mathcal{G},p,q}^\omega(A)$ is well defined.)

An event $A$ is local of $\mathcal{E} \subseteq \mathbb{Z}^d$ if $A \in \mathcal{F}_\mathcal{E}$.

Lemma ("Pushing b.c.") Let $\mathcal{G} = (V,E)$ and $\mathcal{G}' = (V',E')$ subgraphs of $\mathcal{E} \subseteq \mathbb{Z}^d$. For every $A \in \mathcal{F}_{\mathcal{E}}$, increasing.

$$
\phi_{\mathcal{G}',p,q}^\omega(A) \geq \phi_{\mathcal{G},p,q}^\omega(A)
$$

$$
\phi_{\mathcal{G}',p,q}^b(A) \leq \phi_{\mathcal{G},p,q}^w(A)
$$

Proof:

$$
\phi_{\mathcal{G},p,q}^w(A) = \sum_{\psi \in \{0,1\}^E} \phi_{\mathcal{G},p,q}^\omega(A | w_{E \setminus E'} = \psi) \phi_{\mathcal{G}',p,q}^w(w_{E \setminus E'} = \psi)
$$

- DMP: $= \phi_{\mathcal{G}',p,q}^w(A)
- CBC: $\leq \phi_{\mathcal{G}',p,q}^w(A)

$$
\leq \phi_{\mathcal{G}',p,q}^w(A) \sum_{\psi \in \{0,1\}^E} \phi_{\mathcal{G},p,q}^\omega(w_{E \setminus E'} = \psi)
$$
Theorem: Let \( p \in (0,1) \), \( q > 1 \). There exist two probability measures \( \phi^b_{p,q} \) and \( \phi^w_{p,q} \) on \( \mathbb{R} \) characterized by

1. \( \forall A \text{ local event}, \ \mathcal{E}_n = (v_n, E_n) \text{ with } E_n \to \mathbb{R}^d \):
   \[
   \phi^w_{p,q}(A) = \lim_{n \to \infty} \phi^w_{\mathcal{E}_n, p, q}(A),
   \]
   \[
   \phi^b_{p,q}(A) = \lim_{n \to \infty} \phi^b_{\mathcal{E}_n, p, q}(A).
   \]

Proof: Let \( A \) be a local increasing event.

Then for large enough \( \phi^w_{\mathcal{E}_n, p, q}(A) \leq \phi^w_{\mathcal{E}_{n'}, p, q}(A) \)

therefore \( \phi^w_{p,q}(A) = \lim_{n \to \infty} \phi^w_{\mathcal{E}_n, p, q}(A) \) is well defined.

As for Ising, we first extend \( \phi^w_{p,q} \) to local events and then to every events by Kolmogorov's theorem. \( \square \)

Exercises:

1. Prove that \( \phi^b_{p,q} \ll \phi^w_{p,q} \).

2. Prove that \( \phi^b_{p,q} \) and \( \phi^w_{p,q} \) are translation invariant.

3. (difficult) Prove that \( \phi^b_{p,q} \) and \( \phi^w_{p,q} \) are ergodic:

   if \( A \) is a translation invariant event then

   \[ \phi^b_{p,q}(A), \phi^w_{p,q}(A) \in \{0,1\}. \]
2 Uniqueness of the Infinite Volume Measure

Definition:
Let \( q > 1, p \in (0,1) \). We say that there exists a unique infinite-volume FK-measure if \( \phi^b_{p,q} = \phi^w_{p,q} \).

Remark:
It is for Ising model, one can define general infinite-volume FK-measures, and any such measure \( \phi \) would satisfy
\[
\phi^b_{p,q} \ll \phi \ll \phi^w_{p,q}.
\]

Remark: The FK-percolation measure in finite volume can be rewritten as
\[
\phi_{A, p, q}^g(w) = \frac{1}{Z_{A, p, q}} \exp \left( q k_y(w) \sigma(w) - (1-p) \sum_{w \in E} \omega_e \right)
\]
where, the parameter \( p \) plays a "similar role" as the external field \( h \) for Ising.

In particular, the techniques which provides characterizations of uniqueness for Ising in terms of analytic properties in \( h \) apply. Here uniqueness can be understood via analytic properties in \( p \).
Theorem: (Characterization of uniqueness)

Fix \( q \geq 1 \) and \( p \in [0,1] \). The following are equivalent

(i) \( \phi_{p,q}^w = \phi_{p,q}^b \)
(ii) \( \phi_{p,q}^w (w_c = 1) = \phi_{p,q}^b (w_c = 1) \)
(iii) \( p' \rightarrow \phi_{p',q}^w (w_c = 1) \) is continuous at \( p \).

Proof: Similar to Isong's exercise.

Corollary

Fix \( q \geq 1 \). The set \( D_q := \{ p \in [0,1] : \phi_{p,q}^w \neq \phi_{p,q}^b \} \)

is at most countable.

Proof: \( p' \rightarrow \phi_{p',q}^w (w_c = 1) \) is increasing, hence it has at most countably many discontinuity points.

Theorem: (Sufficient condition for uniqueness.)

Let \( 0 \rightarrow \infty \) be the event that the component of 0 is infinite. If \( \phi_{p,q}^w (0 \rightarrow \infty) = 0 \) \( \Rightarrow \) \( \phi_{p,q}^b = \phi_{p,q}^w \).

Proof: Let \( A \) be a local increasing event. Let \( k \geq 1 \) such that \( A \) is measurable with respect to the configuration in \( B_k = \{ -k, -k+1, \ldots, k \} \). We have, for every \( n > k \),

\[
\bigwedge_{n > k} \phi_{B_n, B_{n+1}, B_{n+2}, \ldots, B_{n+k}}^b (A) \leq \phi_{B_n, B_{n+1}, B_{n+2}, \ldots, B_{n+k}}^w (A | B_k \cup \ldots \cup B_n) + \phi_{B_n}^w (0 \leftarrow \infty) \]

stochastic domination
The general idea of the proof is that the event \( \delta_k \to \delta_m \) implies the existence of a "blocking surface" between \( \Omega_k \) and \( \Omega_m \). This blocking surface create Dirichlet boundary conditions for the configuration inside. Define the random set

\[
\mathcal{Y}(w) = \{ \pi \in \Lambda_m : \pi \not\in \delta \Omega_m \text{ in } w \}
\]

Notice that, \( \Lambda_k \subseteq \mathcal{Y}(w) \) if \( \Lambda_k \to \delta \Omega_m \) in \( w \).

The event \( \mathcal{Y}(w) = S \) depends only on the configuration in \( \Lambda_m \setminus S \).

We have

\[
\phi_{B_m, \rho, \eta}^w (A \cap \delta_B \to \delta_m) = \sum_{\delta_k \subseteq \delta_k \cap \delta_B \subseteq \delta_m} \phi_{B_m, \rho, \eta}^w (A \cap \delta_B \to \delta_m) \phi_{\delta_k}^w (\delta \to \delta_m) \]

We have

\[
\phi_{B_m, \rho, \eta}^w (A \cap \delta_B \to \delta_m) \leq \sum_{\delta_k \subseteq \delta_k \cap \delta_B \subseteq \delta_m} \phi_{B_m, \rho, \eta}^w (A \cap \delta_B \to \delta_m) \phi_{\delta_k}^w (\delta \to \delta_m)
\]

Hence

\[
\phi_{B_m, \rho, \eta}^w (A \cap \delta_B \to \delta_m) \leq \phi_{B_{m/B}, \rho, \eta}^w (A)
\]

Plugging this in \( \delta_k \) and letting \( n \) tend to infinity concludes the proof.
As seen in the previous chapter, long-distance cancellations for Ising was encoded by the existence of long path for the corresponding FK-percolation process. With this in mind, it is natural to define the phase transition of FK-percolation by looking at the existence of infinite paths. Before giving the definition of the critical parameter, we give a useful expression for the probability of an infinite path from 0.

**Lemma:** Let $q > 1$, $p \in [0,1)$. For every $G_n \to \mathbb{Z}^d$

$$\phi_{p,q}^w(0 \to \infty) = \lim_{m \to \infty} \phi_{G_m, p, q}^w(0 \to \partial G_m)$$

**Proof:** By "pushing the b.c.", we have for every $k \geq m$

$$\phi_{G_k, p, q}^w(0 \to \partial G_m) \leq \phi_{G_m, p, q}^w(0 \to \partial G_m)$$

Taking first the limit as $k$ tends to infinity, and then the limit as $m$ tends to infinity, we obtain

$$\phi_{p,q}^w(0 \to \infty) \leq \lim_{m \to \infty} \phi_{G_m, p, q}^w(0 \to \partial G_m).$$

On the other hand, we trivially have for $k \geq m$

$$\phi_{G_k, p, q}^w(0 \to \partial G_k) \geq \phi_{G_m, p, q}^w(0 \to \partial G_m)$$

$$\phi_{p,q}^w(0 \to \infty) \geq \lim_{m \to \infty} \phi_{G_m, p, q}^w(0 \to \partial G_m).$$
Taking first the limsup as \( n \) tends to infinity and then the limit as \( k \) tends to infinity, we get:

\[
\Phi_{p,q}^W (0 \to \infty) \geq \limsup_{n \to \infty} \Phi_{Q_n}^W (0 \to \infty)
\]

The expression above directly implies that for fixed \( q > 1 \), \( \Phi_{p,q}^W (0 \to \infty) \) is increasing in \( p \) (as a simple limit of increasing functions).

**Definition:** Fix \( q > 1 \), define the critical parameter for FK-percolation by:

\[
\mathbf{p}_c (q) := \sup \{ p \in [0,1] : \Phi_{p,q}^W (0 \to \infty) = 0 \}
\]

\[
\begin{array}{ccc}
0 & \mathbf{p}_c (q) & 1 \\
\Phi_{p,q}^W (0 \to \infty) = 0 & \Phi_{p,q}^W (0 \to \infty) > 0
\end{array}
\]

**Remark:** The choice of the wired measure is arbitrary here and the critical parameter does not depend on this choice. Indeed, using that the infinite-volume measure is unique for \( p < \mathbf{p}_c (q) \) and the almost everywhere uniqueness for \( p > \mathbf{p}_c (q) \), one can prove that:

\[
\Phi_{p,q}^W (0 \to \infty) = 0 \quad \text{if} \quad p < \mathbf{p}_c (q) \\
\Phi_{p,q}^W (0 \to \infty) > 0 \quad \text{if} \quad p > \mathbf{p}_c (q).
\]
Question: What is the monotonicity of \( p_c(q) \), as a function of the cluster-weight \( q \)?

4. Important Results and Conjectures: An Overview.

4.1. Subcritical phase. \((p < p_c(q))\)

Thm: Fix \( q > 1 \), \( p < p_c(q) \). Then,

(i) [Uniqueness] \( \phi_{p,q}^w = \phi_{p,q}^b \)

(ii) [Exponential Decay] There exists \( c > 0 \) s.t. \((p, q) \) in every \( n \geq 1\)

\[
\phi_{G_n, p, q}^w (0 \to \partial G_n) \leq e^{-cn}
\]

where \( G_n = [-n, n]^2 \)

(i) follows from the Lemma in Section 2.

(ii) is a recent result. Using tools from the theory of randomized algorithms, it is possible to prove that the function \( \Theta_n(p) = \phi_{G_n, p, q}^w (0 \to \partial G_n) \) satisfies

\[
\Theta_n' \geq c \frac{\Theta_n^3}{S_m}
\]

where \( c > 0 \) constant.

and \( S_m = \sum_{k=1}^{m} \Theta_k \)

This differential inequality implies the result.
4.2 Supercritical phase $(p > p_c(q))$

**Theorem:** Fix $q > 1$.

(i) [Mean-field lower bound] \( \exists c > 0 \text{ s.t. } p > p_c(q) \)
\[
\phi_{p, q}^w (0 \rightarrow \infty) \geq c (p - p_c(q))
\]

(ii) [Uniqueness of the infinite cluster]
Let \( \# \in \{\emptyset, w\} \), \( p > p_c(q) \) then
\[
\phi_{p, q}^\# \left[ \exists \text{ a unique infinite cluster in } w \right] = 1
\]

(i) is proved using the differential inequality (*) on page
previous page.

To prove (ii) one first need to show that the two
measures \( \phi_{p, q}^\# , \# \in \{\emptyset, w\} \) are ergodic (translation invariant
events satisfy a 0-1 law). Then, the Burton-Keane
argument (truncation argument) applies, exactly
as for Bernoulli percolation.

**Conjecture:**
\[
\text{If } p > p_c(q) \text{ then } \phi_{p, q}^w = \phi_{p, q}^\emptyset
\]

As we will see later, this conjecture can be proved.
in dimension 2, by duality. Let us here present an argument that suggests why the conjecture should also be true in higher dimensions.

In dim 2, we can prove the following statement.

$$\phi^b_{p,q}(0 \to \infty) \to 0 \Rightarrow \phi^b_{p,q} = \phi^w_{p,q}.$$ 

This implication is very similar to the lemma in Sect 2. It clearly implies the conjecture since $$\phi^b_{p,q}(0 \to \infty) \to 0 \Rightarrow p > p_c(q).$$ The general idea behind the implication above is the following: If under $$\phi^b_{p,q}$$ there exists an infinite cluster, then this infinite cluster should "create" wired boundary conditions around a big box $$\Lambda_m$$, and we expect that

$$\phi^b_{p,q}[\cdot] \subseteq \bigcup \{ \text{finite cluster} \} \Rightarrow \phi^w_{\Lambda_m}[\cdot]$$

since the conditions occur almost surely, this would imply

$$\phi^b_{p,q} = \phi^w_{p,q}.$$ 

This argument can be made rigorous in dimension 2 because the existence of an unique infinite cluster under $$\phi^b_{p,q}$$ implies the existence of a circuit around the origin. Conditioning under such circuit, the configuration inside dominates a wired FK-percolation.
4.3 Critical phase.

The critical behaviour of FK percolation is not well understood, except in few cases.

**Theorem:** In dimension 2, we have:

- If $q \leq 4$, $\phi^w_{P_c(q),q} = \phi^b_{P(q),q}$ and $\phi^w_{P_c(q),q}(0 \to \infty) > 0$

- If $q > 4$, $\phi^w_{P_c(q),q}(0 \to \infty) > 0$

  \[
  \lim_{n \to \infty} \phi^w_{P_c(q),q} \left( \begin{array}{c}
  \lambda \\
  n
  \end{array} \right) = c(\lambda)
  \]

If $q \leq 4$, the model is conjectured to converge to a scaling limit related to SLE processes. This has been proved for $q = 2$ only. In particular, the following convergence is expected to hold.

Conjecture: \( \forall \lambda \in (0, \infty) \exists c = c(\lambda) \in (0,1) \) a.l.
Conjecture: Fix $q \geq 1$. Then there exists $q_c(d) \in (2, \infty)$ a.t.

(i) $\phi^b_{p_c(q), d} (0 \rightarrow \infty) = 0$ for every $q \geq 1$

(ii) $\phi^w_{p_c(q), d} (0 \rightarrow \infty) = \begin{cases} 0 & \text{if } q \leq q_c(d) \\
 > 0 & \text{if } q > q_c(d) \end{cases}$ (continuous phase transition) (discontinuous phase transition)

It is believed that $q_c(d) = 2.2$ if $d \geq 6$

Remark: the conjecture is proved in dimension $d = 2$ with $q_c(2) = 4$.

It is known that

- $\phi^w_{p_c(q), d} (0 \rightarrow \infty) = 0$ for $q = 2$ (Ising case) for every $d \geq 2$
- $\phi^w_{p_c(q), d} (0 \rightarrow \infty) > 0$ for fixed $d \geq 2$ and $q$ large enough.
- $\phi^w_{p_c(q), d} (0 \rightarrow \infty) > 0$ for $q \in \{3, \ldots, \infty\}$ and $d$ large enough.

Let us mention a particular case, when $q = 1$:

Conjecture: Consider Bernoulli percolation $P_p$ on $\mathbb{Z}^d$ at $p = p_c(d)$.
Then $P_{p_c} [0 \rightarrow \infty] = 0$

The conjecture has been proved for $d = 2$ and $d > 4$.
A major open problem is to solve the conjecture for $d = 3$.\}
The results and conjectures are presented on the following diagram (the results are in black, the conjectures in blue).

\[
\begin{align*}
0 & \quad \phi_{p,q}^{\omega} [\omega \sim \omega] = 0 \\
\phi_{p,c(q)}^{\omega} & \quad \phi_{p,q}^{\omega} [\omega \sim \omega] > 0 \\
[\text{UNIQUENESS}] & \quad \phi_{p,q}^{\omega} = \phi_{p,q}^0 \\
[\text{UNIQUENESS}] & \quad \phi_{p,c(q)}^{\omega} = 1 \\
[\text{UNIQUE CLUSTER}] & \quad \phi_{p,c(q)}^{\omega} [\omega \sim \omega] = 0 \\
[\text{UNIQUE CLUSTER}] & \quad \phi_{p,c(q)}^{\omega} [\omega \sim \omega] > 0
\end{align*}
\]

**Expected behavior of the percolation density**

- Continuous phase transition \((q < q_c(d))\)
- Discontinuous phase transition \((q > q_c(d))\)
\textbf{5) Consequences for Ising Model (}q = 2\textbf{)}

**Thm:** Let $\beta_c$ be the critical inverse temperature for Ising Model on $\mathbb{Z}^d$.

\[ p_c(\beta) = 1 - e^{-2\beta_c}. \]

For all $\beta < \beta_c$, we have:

(i) $\langle \sigma_0 \sigma_x \rangle_{\beta}^\infty \leq e^{-c|x|}$

(ii) $\langle \sigma_0 \rangle_{\Lambda^*_n, \beta} \leq e^{-c m}$ if $\Lambda_m = [-m, m)$

**Proof:** Let $p = 1 - e^{-2\beta}$, then

\[ \langle \sigma_0 \rangle_{\beta}^\infty = \lim_{\Lambda^*_n, \beta} \langle \sigma_0 \rangle_{\Lambda^*_n, \beta}. \]

\[ = \lim_{\Lambda^*_n, \beta} \phi_{\Lambda^*_n, p^2}^{\infty} (o \rightarrow \infty) \] (\( \Lambda_m = (\overline{\Lambda_m}, E_{\Lambda_m}) \))

\[ = \phi_{p^2}^{\infty} (o \rightarrow \infty) \]

This implies directly $p_c(2) = 1 - e^{-2\beta_c}$.

To prove (i), observe that

\[ \langle \sigma_0 \sigma_x \rangle_{\beta}^\infty = \phi_{p^2}^{\infty} (o \rightarrow x) \]

\[ \leq \phi_{p^2}^{\infty} (o \rightarrow d B_{|x|}) \]

\[ \leq \phi_{B_{|x|}, p^2}^{\infty} (o \rightarrow d B_{|x|}) \]

\[ \leq e^{-c|x|}. \]
1. FK-PERCOLATION ON A FINITE PLANE GRAPH.

Let $G = (V, E)$ be a finite connected plane graph. A vertex is a point in $\mathbb{R}^2$, an edge is an arc between two vertices. Any edge intersects no vertex and no other edge (except at its extremities).

![Diagram of a plane graph $G = (V, E)$]

A face of $G$ is a connected component of $\mathbb{R}^2 \setminus G$.

A plane graph $G^*$ is a dual of $G$ if each face of $G$ contains exactly one vertex of $G^*$, each vertex of $G$ is contained in exactly one face of $G^*$, and each edge $e$ of $G$ is crossed by exactly one edge $e^*$ of $G^*$ at one point.

![Diagram of a plane graph $G$ and one of its dual (in blue)]
As before, we can define FK percolation on $\mathbb{Z}^d$ by setting for $p \in (0, 1)$ and $q > 0$, for any $w \in \{0, 1\}^E$:

$$\phi_{q,p,q}(w) = \frac{1}{Z_{q,p,q}} \prod_{(i,j) \in \partial w} (1-p) c(i,j) q k(i,j)$$

where $k(i,j)$ is the number of connected components of $w$, identified to the subgraph $(V, \{e : w(e) = 1\})$.

Remark: note that here there is no boundary condition.

The dual configuration $w^* \in \{0, 1\}^E$ of $w$ is defined by

$$\forall e \in E \quad w^*(e^*) = 1 - w(e)$$

**Theorem.** For every $w \in \{0, 1\}^E$, we have

$$\phi_{q,p,q}(w) = \phi_{q^*,p^*,q^*}(w^*)$$

where

$$\frac{p^*}{1-p^*} = \frac{q(1-p)}{p}$$
graph of $p \rightarrow p^* = \frac{(1-p)q}{p + (1-p)q}$ for different $q$

Note that $psd := \frac{Nq^2}{1+q}$ is the unique fixed point of the mapping $p \rightarrow p^*$.

**Proof.** We write $Cte$ for a constant independent of $w$ (it may depend on $y, p, q$, and change from line to line). Note that

$$\phi_{y/p/q}(w) = Cte \cdot \left(\frac{p}{1-p}\right)^{\sigma(w)} q^k(w)$$

In order to prove (4) we express $\sigma(w)$ and $k(w)$ in terms of $\sigma(w^*)$ and $k(w^*)$. First, we have

$$\sigma(w) = 18\ | - \sigma(w^*)$$

In order to express $k(w)$, we apply Euler formula to the graph $G_w = (V, \{ e \mid w(e) = 1 \})$ and having $f(w^*)$ the number of faces of $G_w$, we have

$$|V| - \sigma(w) + f(w) = 1 + k(w).$$

([vertices] - [edges] + [faces] = 1 connected components)
We claim that

$$f(w^*) = k(w)$$ (2)

This statement is rather intuitive since the clusters of $w$ are exactly included in the faces of $w^*$ (see the picture at the beginning of the chapter). We defer the rigorous proof of (2) to the end of the proof. Plugging (2) in Euler formula we get

$$k(w) = 1 + k(w^*) + \sigma(w^*) - |V^*|$$

Hence

$$\Phi_{g, p, q} (w) = \frac{1}{Cte} \left( \frac{p}{1-p} \right)^{\sigma(w^*)} \left(1 - |V^*| + k(w^*) + \sigma(w^*)\right)$$

$$= \frac{1}{Cte} \left( \frac{q(1-p)}{p} \right)^{\sigma(w^*)} k(w^*)$$

$$= \frac{1}{Cte} \left( \frac{p^*}{1-p^*} \right)^{\sigma(w^*)} k(w^*)$$

$$= \Phi_{g, p^*, q} (w^*)$$

It remains to prove (2). Since the edges of $G_w$ cannot intersect the edges of $G_{w^*}$, the clusters of $G_w$ must be included in the faces of $G_{w^*}$, hence

$$f(w^*) \leq k(w)$$
Equivalently, we have $f(w) \leq k(w^*)$

By Euler formula (applied to $G_w$ and $G_{w^*}$), this implies

$$1 + k(w) - \sigma(w) - |V| \leq |V^*| - \tau(w^*) + f(w^*) - 1$$

Since $|V| = (\sigma(w) + \tau(w^*)) + |V^*| = 2$ (by Euler formula applied to the graph $G$), we finally get

$$k(w) \leq f(w^*)$$

2. Duality on the Square Lattice

Let $(\mathbb{Z}^2)^*, (E^*)^*$ be the graph obtained by translating $(\mathbb{Z}^2, E)$ by vector $(\frac{1}{2}, \frac{1}{2})$

Each edge of $E^2$ crosses a unique edge of $(E^2)^*$, denoted $e^*$. Given $w \in \{0, 1\}^{E^2}$ we define $w^* \in \{0, 1\}^{(E^2)^*}$ by setting

$$w^*(e^*) = 1 - w(e)$$
Thm: Let $X$ be a random configuration with law $\phi^w_{p,q}$.

Then $X^*$ has law $\phi_{p,q}^0$ on $(\mathbb{Z}^2)^*$

Proof: Let $B_n = \{-n, \ldots, n\}$ and $G_n$ the subgraph of $\mathbb{Z}^2$
induced by $B^-_n$. Let $G_m$ be the plane graph obtained
from $G_n$, by contracting all the vertices of $G_n$ into one vertex.

The plane graph $G_n$
and its dual $G^*_n$ (in blue)

Notice that the subgraph $G^*_n$ of $(\mathbb{Z}^2)^*$ induced by
$\{-n+\frac{1}{2}, \ldots, n-\frac{1}{2}\}^2$ is a dual of $G_n$. Furthermore
a configuration $w$ on $G_n$ can be seen as a configuration
on $G^*_n$, and

$\phi^w_{G_n, B_n}(w) = \phi^w_{G_m, B_m}(w) = \phi^0_{G^*_n, B_m}(w)$
Hence, in every local event $A$

$$\phi_{p,r,q}^w(A) = \phi_{r,q}^p(A^*)$$

where $A^* = \{w^* : w \in A\}$.

Hence in every local event $X^*$

$$P[X^* \in A^*] = P[X \in A]$$

$$= \phi_{r,q}^w(A)$$

$$= \phi_{r,q}^p(A^*)$$

Remark: In general it is possible to define a notion of a planar b.c. $p$ and their dual $p^*$, in such a way that in every $w$ configuration on $G_n$

$$\phi_{G_n,p,q}(w) = \phi_{G_n,p^*,q}(w^*)$$

In the proof above we proved it for $p = w$ and $p^* = p$.

Exercise: Let

$$G_n = \begin{array}{c}
\begin{tikzpicture}
\fill[white] (0,0) rectangle (2,2);
\draw (0,0) -- (2,2);
\draw (0,2) -- (2,0);
\draw (0.5,0) -- (0.5,2);
\draw (1,0) -- (1,2);
\draw (1.5,0) -- (1.5,2);
\draw (0,0.5) -- (2,0.5);
\draw (0,1) -- (2,1);
\draw (0,1.5) -- (2,1.5);
\draw (0.2,0.2) -- (0.2,0.8);
\draw (0.8,0.2) -- (0.8,0.8);
\draw (0.2,1.2) -- (0.2,1.8);
\draw (0.8,1.2) -- (0.8,1.8);
\draw (1.2,0.2) -- (1.2,0.8);
\draw (1.8,0.2) -- (1.8,0.8);
\draw (1.2,1.2) -- (1.2,1.8);
\draw (1.8,1.2) -- (1.8,1.8);
\end{tikzpicture}
\end{array}$$

Consider the b.c. mix: wired on the left and right, and free on top and bottom. Prove that

$$\phi_{mix}^{G_n,(B)} = \frac{Nq}{1 + Nq}$$
3) Computation of $p_c(\mathcal{G})$

Let $G_m$ be the graph illustrated in the exercise above. Consider the event that there exists a path from left to right in $G_m$ in the configuration $\omega$.

**Lemma:** Fix $q \geq 1$

1) For $p < p_c(q)$, we have $\lim_{m \to \infty} \phi_{p,q}^{\omega} \left[ \begin{array}{c} \mathcal{G} \\ \end{array} \right] \to 0$.

2) For $p > p_c(q)$, we have $\lim_{m \to \infty} \phi_{p,q}^{\omega} \left[ \begin{array}{c} \mathcal{G} \\ \end{array} \right] \to 1$.

**Proof:**

1) Let $L \subseteq G_m$ be the left side of $G_m$. If there is a path from left to right in $G_m$, then there must exist a point $x \in L$ that is connected to distance $m$ around it. Therefore, by translational invariance,

$$\phi_{p,q}^{\omega} \left[ \begin{array}{c} \mathcal{G} \\ \end{array} \right] \leq |L| \times \phi_{p,q}^{\omega} \left[ \begin{array}{c} \mathcal{G} \\ \end{array} \right]_{x \to \infty} \leq (2m+1) e^{-cm} \to 0 \quad m \to \infty.$$
2) Consider the graph \( \Lambda_n = \sum_{i=1}^{4} \Lambda_i \) left right top bottom.

Let \( \varepsilon > 0 \). Pick \( m \) large enough such that
\[
\phi^b_{\rho, \lambda} \left[ \Lambda_m \rightarrow \infty \right] \geq 1 - \varepsilon^4.
\]
(possible because \( \phi^b_{\rho, \lambda} (0 \rightarrow \infty) = \infty \))

In particular, for every \( m \geq m \)
\[
\phi^b_{\rho, \lambda} \left[ \Lambda_m \rightarrow \exists \Lambda_n \right] \geq 1 - \varepsilon^4
\]

i.e.
\[
\phi^b_{\rho, \lambda} \left[ \bigcup_{i=1}^{4} (\Lambda_m \leftrightarrow S_i) \right] \geq 1 - \varepsilon^4.
\]

This implies for every \( i \),
\[
\phi^b_{\rho, \lambda} \left[ \Lambda_m \leftrightarrow S_i \right] \geq 1 - \varepsilon
\]

(indeed, by FKG, inequality and symmetry, we have
\[
\phi^b_{\rho, \lambda} \left[ \Lambda_m \leftrightarrow S_1 \right] \leq \phi^b_{\rho, \lambda} \left[ \Lambda_m \leftrightarrow \exists \Lambda_n \right] \leq \varepsilon^4
\]
Now \( \phi_{r,q}^{m} (\text{graph}) \geq \phi_{r,q}^{m} (\text{square}) - \phi_{r,q}^{m} (\text{triangle}) \geq 2(1-c)^2 \)

Hence, \( \liminf_{m \to \infty} \phi_{r,q}^{m} (\text{graph}) \geq (1-c)^2 \)

\[ \begin{align*}
\text{Thm: For FK-percolation on } \mathbb{Z}^2, & \text{ we have} \\
\text{we have} & \\
\lim_{m \to \infty} & = \frac{\sqrt{q}}{1+\sqrt{q}}
\end{align*} \]

Proof: \( P_{sd} = \frac{\sqrt{q}}{1+\sqrt{q}} \) is such that \( P_{sd}^* = P_{sd} \)

Let \( X \sim \phi_{p,q}^{\infty} \) (Hence \( X^* \sim \phi_{p,q}^{\infty} \))

We have

\[ \begin{align*}
\text{sym} & + P_{sd} = P_{sd} \\
\text{we have} & \\
\text{we have} & \\
\phi_{p,q}^{\infty} (\text{graph}) + \phi_{p,q}^{\infty} (\text{square}) & = 1
\end{align*} \]
Since \( \phi_{p, q} \left( \begin{array}{c}
\end{array} \right) \geq \phi_{p, q} \left( \begin{array}{c}
\end{array} \right) \)

we obtain, for every \( m \)

\[ \phi_{p, q} \left( \begin{array}{c}
\end{array} \right) \geq \frac{1}{2} \quad \text{and} \quad \phi_{p, q} \left( \begin{array}{c}
\end{array} \right) \leq \frac{1}{2} \]

and therefore, by the lemma \( \text{Psd} = \text{Pc} \).

\[ \text{Corollary:} \]

For Ising model on the square lattice, we have

\[ \beta_c = \frac{1}{2} \log \left( 1 + \sqrt{2} \right) \]