PERCOLATION
&
ISING MODEL —

References

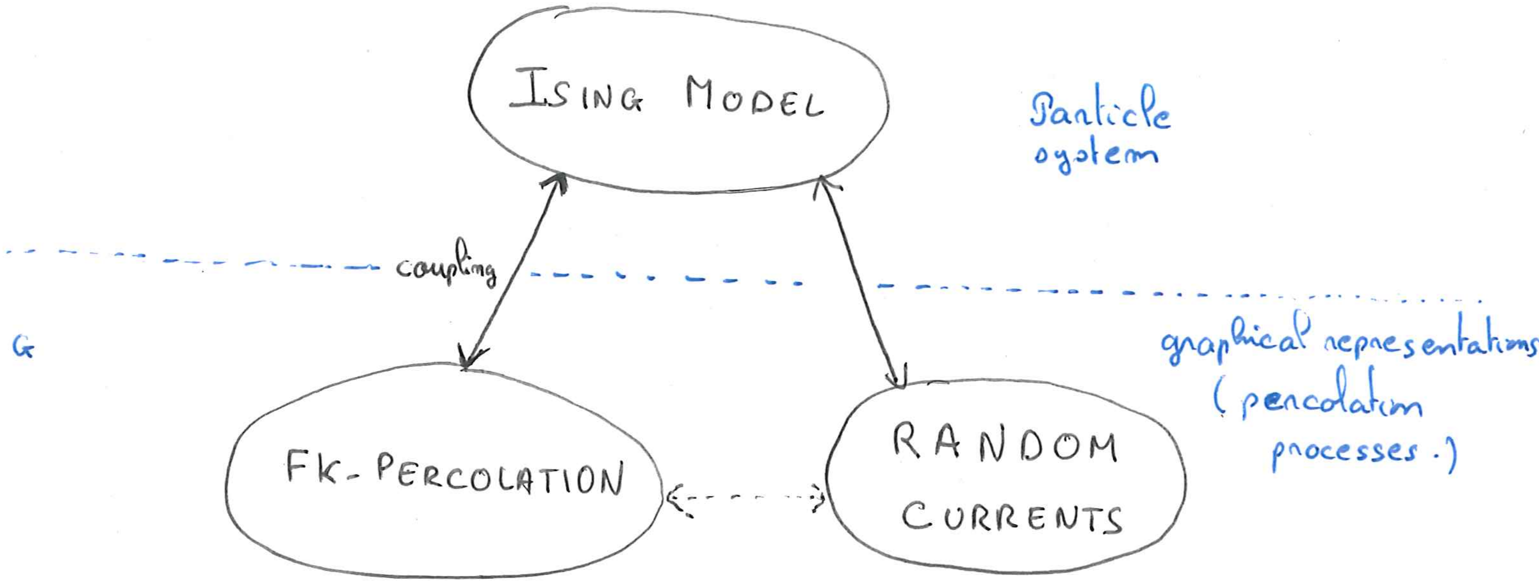
[1] Le modèle d'Ising (In French)
Yvan Velenik http://www.unige.ch/math/folks/velenik/cours.html.

[2] Statistical mechanics of Lattice systems.
S. Friedli & Y. Velenik http://www.unige.ch/math/folks/velenik/smbook

Presentation.

In this course, we define and discuss the main properties of three models in statistical physics: the Ising model and two of its graphical representations, FK-percolation and the random currents. For specific choices of the parameters, these three models can be coupled together. For example, an Ising configuration can be sampled from a FK-percolation configuration, and conversely. These correspondences are very useful and lead to elegant reasonings in the mathematical study of these models: a delicate question for the Ising model may translate into a simple one for FK-percolation.

Our first goal is to provide the mathematical background on these three models. Also, we present some recent develo-pements in the theory, focusing on arguments exploiting the dictionary between the models.



PLAN:

PART 1 : ISING MODEL

PART 2 : FK - PERCOLATION

PART 3 : RANDOM CURRENTS _

PART 1

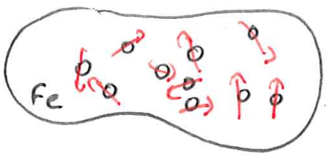
ISING MODEL —

INTRODUCTION.

1 PHYSICAL MOTIVATION: PARA/FERRO MAGNETIC PHASE TRANSITION.

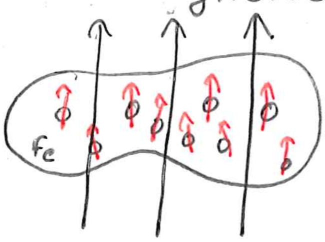
Introduced by Lenz in 1920 in view of a theoritical understanding of the para/ferromagnetic phase transition, the model was named after Ising (Lenz's student) who studied the one-dimensional version of the model in his phd thesis (1925). In this section we give a very brief description of the para/ferro magnetic phase transition.

- Consider a piece of Iron at temperature T, without external field.



the Iron is not magnetized

- Add a magnetic field.

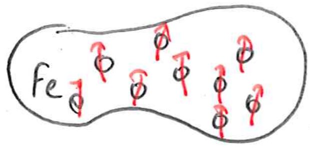


the Iron gets magnetized in the same direction of the field.

- Remove the magnetic field.

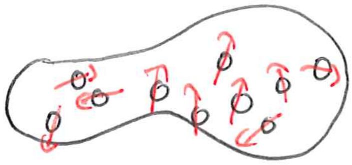
2 cases

if $T<T_c$ →



The Iron remains magnetized (Ferromagnetic behavior)

$T>T_c$ →

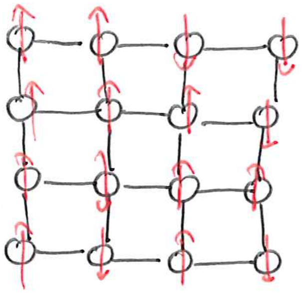


The Iron loses its magnetization (Paramagnetic behaviour)

$T_c(Fe) = 1034\ K$ Curie temperature (Pierre Curie 1895)

- A precise description of the system at a microscopic scale is very difficult because the number of atoms ($\simeq N_A \approx 10^{23}$) is huge.
- In statistical physics, the state of the system is described by a random variable $X = (X_i)_{1 \leq i \leq n}$ where X_i represents the state of the i-th particle.

First model: Bernoulli site percolation.



particles: $\Lambda = \{-n, \dots, n\}^2$

state: $X = (X_i)_{i \in \Lambda} \in \{-1, +1\}^\Lambda$ i
i.i.d. with $\mathbb{P}[X_i = +1] = \frac{e^h}{e^h + e^{-h}}$ and $\mathbb{P}[X_i = -1] = \frac{e^{-h}}{e^h + e^{-h}}$

$X_i = +1$ correspond to a particle with magnetic momentum pointing up $\Uparrow$
and $X_i = -1$ a particle pointing down $\Downarrow$

$h > 0$ corresponds to a positive field $\uparrow$
$h < 0$ ———— negative field $\downarrow$

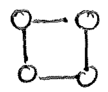
In this simple model the particles have random states which tend to align with the field, but this model is "too naive" to explain the ferromagnetic behaviour.

Ising Model: a state is also described by a random variable $X \in \{-1, 1\}^\Lambda$ but it has the following properties.

- X_i tend to align with the field. (as in the model above)
- X_i tend to align with X_j when i and j are neighbouring particles.

2 ISING MODEL ON A SQUARE WITHOUT MAGNETIC FIELD.

Before presenting the Ising measure in full generality we begin with a very simple example.

We consider the graph $G =$  with vertex set $\Lambda = \{0,1\}^2$

A spin configuration is an element $\sigma \in \{-1,1\}^\Lambda$

Fix $\beta > 0$, and define the energy of σ by

$$\boxed{H_\beta(\sigma) := -\beta \sum_{i,j \text{ neighbours}} \sigma_i \sigma_j}$$

Notice that $\sigma_i \sigma_j = -2 \times \mathbb{1}_{\sigma_i \neq \sigma_j} + 1$

so $$H_\beta(\sigma) = 2 \cdot \beta \cdot \sum_{i,j \text{ neighbours}} \mathbb{1}_{\sigma_i \neq \sigma_j} - 4$$

$\rightarrow$ the energy is large when the number of disagreeing neighbours is large

Define the probability of a configuration

$$\boxed{\mu_\beta(\sigma) := \frac{1}{Z_\beta} e^{-H_\beta(\sigma)}} \quad \text{where } Z_\beta = \sum_{\sigma \in \{-1,1\}^\Lambda} e^{-H_\beta(\sigma)}$$

- Z_β is the partition function. It is defined in such a way that μ_β is a probability measure.

- Rk: If σ has a "large" energy $(H_\beta(\sigma)$ then $\mu_\beta(\sigma)$ is "small".

In this simple case we can determine the measure μ_β.
For each of the 16 configurations σ, we can compute $\mu_\beta(\sigma)$.

configurations σ	energy	probability $\mu_\beta(\sigma)$ $\beta \searrow 0$	$0 < \beta < \infty$	$\beta \to \infty$
$\begin{smallmatrix}+&+\\+&+\end{smallmatrix}$ $\quad$ $\begin{smallmatrix}-&-\\-&-\end{smallmatrix}$	-4β	$\frac{1}{16}$	$\frac{1}{Z_\beta} e^{+4\beta}$	$\frac{1}{2}$
$\begin{smallmatrix}+&+\\+&-\end{smallmatrix} \times 4$ $\quad$ $\begin{smallmatrix}-&-\\-&+\end{smallmatrix} \times 4$ $\quad$ $\begin{smallmatrix}-&-\\+&+\end{smallmatrix} \times 4$	0	$\frac{1}{16}$	$\frac{1}{Z_\beta}$	0
$\begin{smallmatrix}+&-\\-&+\end{smallmatrix}$ $\quad$ $\begin{smallmatrix}-&+\\+&-\end{smallmatrix}$	$+4\beta$	$\frac{1}{16}$	$\frac{1}{Z_\beta} e^{-4\beta}$	0

where $Z_\beta = 2e^{4\beta} + 12 + 2e^{-4\beta}$.

We can see that the spins of $(0,0)$ and $(1,1)$ "interact" when $\beta > 0$

	$\beta \to 0$	$\beta \to \infty$
$$\mu_\beta\left[\sigma((0,0)) = \sigma((1,1))\right] = \frac{2e^{4\beta} + 4 + 2e^{-4\beta}}{2e^{4\beta} + 12 + 2e^{-4\beta}}$$	$\frac{1}{2}$	$\to 1$
	the spin are independent.	the spin are highly correlated

We also observe the following convergences.

$$\mu_\beta \xrightarrow[\beta \to 0]{} \mu_0 \quad \text{(uniform)}$$

$$\mu_\beta \xrightarrow[\beta \to \infty]{} \frac{1}{2}\delta_{+1} + \frac{1}{2}\delta_{-1}$$

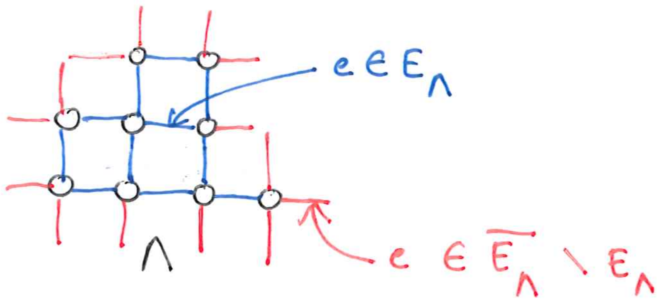
CHAPTER 1 : ISING MODEL IN FINITE VOLUME

In this chapter we fix $d \geq 1$.

Notation: We write $\Lambda \subset\subset \mathbb{Z}^d$ if $\Lambda \subset \mathbb{Z}^d$ and Λ finite.

Let $E_\Lambda = \{ \{i,j\} \subset \Lambda \text{ s.t. } \|i-j\|_1 = 1 \}$

$\overline{E}_\Lambda = \{ \{i,j\} \subset \mathbb{Z}^d \text{ s.t. } \|i-j\|_1 = 1 \text{ and } \{i,j\} \cap \Lambda \neq \emptyset \}$.



The set of spin configurations in Λ is defined by

$$\Omega_\Lambda = \{-1,+1\}^\Lambda = \{\sigma : \Lambda \to \{-1,+1\}\}$$

1 FREE BOUNDARY CONDITIONS

Def: Let $\Lambda \subset\subset \mathbb{Z}^d$, $\beta > 0$, $h \in \mathbb{R}$. The Ising measure in Λ with free boundary conditions (b.c.), inverse temperature β and external field h is defined by

$$\boxed{\forall \sigma \in \Omega_\Lambda \quad \mu^\emptyset_{\Lambda,\beta,h}(\sigma) = \frac{1}{Z^\emptyset_{\beta,h}} e^{-H^\emptyset_{\Lambda,\beta,h}(\sigma)}}$$

where
- $H^\emptyset_{\Lambda,\beta,h}(\sigma) = -\beta \sum_{\{i,j\} \in E_\Lambda} \sigma_i \sigma_j - h \sum_{i \in \Lambda} \sigma_i$
- $Z^\emptyset_{\Lambda,\beta,h} = \sum_{\sigma \in \Omega_\Lambda} e^{-H^\emptyset_{\Lambda,\beta,h}(\sigma)}$

Terminology:

- The function $H^{\emptyset}_{\Lambda,\beta,h}$ is called the Hamiltonian, and $H^{\emptyset}_{\Lambda,\beta,h}(\sigma)$ is the energy of σ.

  If a configuration has a "large" energy, then its probability is "low": in other words, the measure μ_β favors the states with low energy.

- The quantity $Z^{\emptyset}_{\Lambda,\beta,h}$ is the partition function.

Effect of the inverse temperature β.

- When β is large (low temperature), the system favors strongly the states σ with $\sum_{\{i,j\}\in E} \sigma_i\sigma_j$ small, i.e. states with a small number of disagreeing neighbours.

- When β is small (high temperature), the interaction between neighbours is small and the random state looks more like Bernoulli iid variable.

Effect of the field h.

- When $h=0$, the system is invariant under spin-flip: σ and $-\sigma$ have the same probability.
- When $h>0$ (resp <0) the system favors states whith more + spin (resp. − spins).

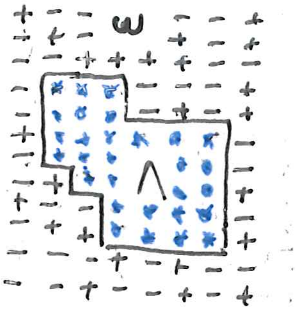
Remark The notation $\emptyset$ is used to symbolize free b.c.

"the particles in Λ do not interact with the particles in Λ^c."

2 CONFIGURATIONAL BONDARY CONDITIONS -

Notation: We write $\Omega := \{-1,+1\}^{\mathbb{Z}^d}$.

For $\omega \in \Omega$ and $\Lambda \subset\subset \mathbb{Z}^d$, we consider the set of configurations $\Omega_{\Lambda,\omega} := \{\sigma \in \Omega : \forall i \in \mathbb{Z}^d \setminus \Lambda \ \sigma_i = \omega_i\}$



A configuration in $\Omega_{\Lambda,\omega}$

- is equal to ω outside Λ
- can take any value inside Λ ($*$)

Remark For ω fixed the projection $\pi_{\Lambda,\omega} : \Omega \longrightarrow \Omega$

Def.: Let $\omega \in \Omega$, $\beta \geq 0$, $h \in \mathbb{R}$. The <u>Ising measure</u> in Λ with b.c. ω, external field h, at inverse temperature β, is defined by

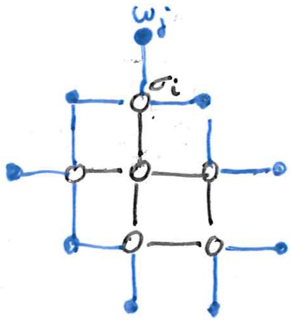
$$\forall \sigma \in \Omega^\omega_\Lambda \quad \boxed{\mu^\omega_{\Lambda,\beta,h}(\sigma) = \frac{1}{Z_{\Lambda,\beta,h}} e^{-H^\omega_{\Lambda,\beta,h}(\sigma)}}$$

where $\cdot\, H^\omega_{\Lambda,\beta,h}(\sigma) = -\beta \sum_{\{i,j\}\in E_\Lambda} \sigma_i\sigma_j - h\sum_{i\in\Lambda}\sigma_i$

$$\cdot\ Z^\omega_{\Lambda,\beta,h}(\sigma) = \sum_{\sigma\in\Omega_{\Lambda,\omega}} e^{-H^\omega_{\Lambda,\beta,h}(\sigma)}$$

<u>Remark</u>: • The energy $H^{\omega}_{\Lambda,\beta,h}(\sigma)$ implicitely depends on ω, due to sum over the edges at the boundary of Λ which have an extremity j in $\mathbb{Z}^d \setminus \Lambda$ where the spin is given by ω_j:

$$\sum_{\{i,j\}\in \overline{E}_\Lambda} \sigma_i \sigma_j = \sum_{\{i,j\}\in E_\Lambda} \sigma_i \sigma_j + \sum_{\substack{\{i,j\}\in \overline{E}_\Lambda \setminus E_\Lambda \\ j\in \mathbb{Z}^d \setminus \Lambda}} \sigma_i \, \omega_j$$



In black : the set Λ and its internal edges E_Λ

In blue : the boundary edges of $\overline{E}_\Lambda \setminus E_\Lambda$

• In particular, we can see that the bondary condition ω plays a role only at the boundary of Λ. Namely, $\mu^{\omega}_{\Lambda,\beta,h}$ and $\mu^{\omega'}_{\Lambda,\beta,h}$ induce the same law on Ω_Λ if

$$\omega_j = \omega_j' \quad \forall\, j \in \partial^{ext}\Lambda = \{ j \in \mathbb{Z}^d \setminus \Lambda : \exists i \in \Lambda \ \|i-j\|_1 = 1 \}$$

<u>Notation</u> : The boundary conditions "all plus" and "all minus" will be important. We write

$$\mu^{+}_{\Lambda,\beta,h} := \mu^{\omega^+}_{\Lambda,\beta,h} \quad \text{where } \omega^+_i = +1 \ \forall i$$

$$\mu^{-}_{\Lambda,\beta,h} := \mu^{\omega^-}_{\Lambda,\beta,h} \quad \text{where } \omega^-_i = -1 \ \forall i$$

Proposition (compatibility between b.c.):

Let $\Delta \subset \Lambda \subset\subset \mathbb{Z}^d$. Let $\omega \in \Omega$, $\beta \geq 0$, $h \in \mathbb{R}$. Then

$$\boxed{\forall \omega' \in \Omega_{\Lambda,\omega} \quad \mu^{\omega}_{\Lambda,\beta,h}(\cdot \mid \forall i \in \Lambda \setminus \Delta \ \sigma_i = \omega_i) = \mu^{\omega'}_{\Delta,\beta,h}(\cdot)}$$

Pf: Let $\eta \in \Omega^{\omega}_{\Lambda}$. We have, by definition,

$$\mu^{\omega}_{\Lambda,\beta,h}(\eta \mid \forall i \in \Lambda \setminus \Delta \ \sigma_i = \omega'_i) = \begin{cases} 0 & \text{if } \eta \notin \Omega_{\Lambda,\omega'} \\ \dfrac{e^{-H^{\omega}_{\Lambda,\beta,h}(\eta)}}{Z^{\omega}_{\Lambda,\beta,h}} & \text{if } \eta \in \Omega_{\Lambda,\omega'} \end{cases}$$

Observe that for every $\sigma \in \Omega_{\Lambda,\omega'}$

$$H^{\omega}_{\Lambda,\beta,h}(\sigma) = \underbrace{-\beta \sum_{\{i,j\} \in E_\Lambda \setminus E_\Delta} \omega'_i \omega'_j - h \sum_{i \in \Lambda \setminus \Delta} \omega'_i}_{=: C(\omega')} + H^{\omega'}_{\Delta,\beta,h}(\sigma)$$

where $C(\omega')$ does not depend on σ.

Hence, if $\eta \in \Omega_{\Lambda,\omega'}$,

$$\frac{e^{-H^{\omega}_{\Lambda,\beta,h}(\eta)}}{Z^{\omega}_{\Lambda,\beta,h}} = \frac{e^{-C(\omega')} \times e^{-H^{\omega'}_{\Delta,\beta,h}(\eta)}}{\sum_{\sigma \in \Omega_{\Lambda,\omega'}} e^{-C(\omega')} e^{-H^{\omega'}_{\Delta,\beta,h}(\sigma)}}$$

$$= \mu^{\omega'}_{\Delta,\beta,h}(\eta) \qquad \blacksquare$$

Notation: For every b.c. $\omega \in \Omega$, we write $\langle \cdot \rangle^{\omega}_{\Lambda,\beta,h}$ for the expectation relative to $\mu^{\omega}_{\Lambda,\beta,h}$. That is

$$\forall f : \Omega^{\omega}_{\Lambda} \to \mathbb{R} \quad \boxed{\langle f \rangle^{\omega}_{\Lambda,\beta,h} := \sum_{\sigma \in \Omega_{\Lambda,\omega}} f(\sigma)\, \mu^{\omega}_{\Lambda,\beta,h}(\sigma)}$$

Equivalently, for the free b.c., we write

$$\forall f : \Omega_{\Lambda} \to \mathbb{R} \quad \boxed{\langle f \rangle^{\emptyset}_{\Lambda,\beta,h} := \sum_{\sigma \in \Omega_{\Lambda}} f(\sigma)\, \mu^{\emptyset}_{\Lambda,\beta,h}(\sigma)}$$

3 CORRELATION INEQUALITIES

3.1 GKS* INEQUALITIES

[* Griffiths, Kelly, Sherman]

Notation: For $A \subset\subset \mathbb{Z}^d$, for $\sigma \in \Omega$, we write

$$\boxed{\sigma_A := \prod_{i \in A} \sigma_i}$$

(σ_A can be seen as a function $\sigma_A : \Omega^{\omega}_{\Lambda} \to \{-1,1\}$ or $\sigma_A : \Omega_{\Lambda} \to \{-1,1\}$ when $A \subset \Lambda$)

Thm: (GKS inequalities)

Let $\Lambda \subset\subset \mathbb{Z}^d$, $h \geq 0$, $\beta \geq 0$, $\# \in \{\emptyset, +\}$. Then

- $\forall A \subset \Lambda \quad \langle \sigma_A \rangle^{\#}_{\Lambda,\beta,h} \geq 0$
- $\forall A, B \subset \Lambda \quad \langle \sigma_A \sigma_B \rangle^{\#}_{\Lambda,\beta,h} \geq \langle \sigma_A \rangle^{\#}_{\Lambda,\beta,h} \langle \sigma_B \rangle^{\#}_{\Lambda,\beta,h}$.

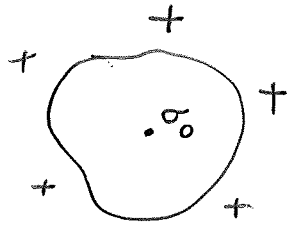
⚠ valid only for the b.c. $\emptyset$ and $+$ and for $h \geq 0$!

We will prove the GKS inequalities later, using the random-current representation of Ising model.

The GKS inequalities confirm the intuition that when the external action on the system "favors" the + spin ($h \geq 0$ and/or + boundary conditions), then the spins σ_i in the system "want" to align with + ($\langle \sigma_i \rangle \geq 0$) and also they "want" to align together ($\langle \sigma_i \sigma_j \rangle \geq 0$). This effect gets reinforced as β increases (see the corollary below)

Application.

For $A = \{0\}$ we get $\langle \sigma_0 \rangle^{+}_{\Lambda,\beta,h} \geq 0$



(The + b.c. and the positive field $h \geq 0$ influence positively the spin at 0)

The GKS inequalities provide monotonicity properties in β:

Corollary: (monotonicity of the magnetization in β.)

Let $\Lambda \subset\subset \mathbb{Z}^d$, $h \geq 0$, $\# \in \{\emptyset, +\}$

$$\boxed{0 \leq \beta \leq \beta' \Rightarrow \langle \sigma_0 \rangle^{\#}_{\Lambda,\beta,h} \leq \langle \sigma_0 \rangle^{\#}_{\Lambda,\beta',h}}$$

⚠ valid only for the b.c. $\emptyset$ and + and $h \geq 0$.

Pf: (for the + b.c.)

The quantity $\langle \sigma_0 \rangle^+_{\Lambda,\beta,h} = \dfrac{\sum_{\sigma \in \Omega^+_\Lambda} \sigma_0 \, e^{-H^+_{\Lambda,\beta,h}(\sigma)}}{\sum_{\sigma \in \Omega^+_\Lambda} e^{-H^+_{\Lambda,\beta,h}(\sigma)}}$

is differentiable in β and

$$\frac{d}{d\beta} \langle \sigma_0 \rangle^+_{\Lambda,\beta,h} = \frac{\sum_\sigma \sum_{\{i,j\} \in \bar{E}_\Lambda} \sigma_i \sigma_j \sigma_0 \, e^{-H^+_{\Lambda,\beta,h}(\sigma)}}{\sum_{\sigma \in \Omega^+_\Lambda} e^{-H^+_{\Lambda,\beta,h}(\sigma)}}$$

$$- \frac{\sum_\sigma \sigma_0 e^{-H^+_{\Lambda,\beta,h}(\sigma)} \times \sum_\sigma \sum_{\{i,j\} \in \bar{E}_\Lambda} \sigma_i \sigma_j \, e^{-H^+_{\Lambda,\beta,h}(\sigma)}}{\sum_{\sigma \in \Omega^+_\Lambda} e^{-H^+_{\Lambda,\beta,h}(\sigma)}}$$

$$= \sum_{\{i,j\} \in \bar{E}_\Lambda} \langle \sigma_i \sigma_j \sigma_0 \rangle^+_{\Lambda,\beta,h} - \langle \sigma_{ij} \rangle^+_{\Lambda,\beta,h} \langle \sigma_0 \rangle^+_{\Lambda,\beta,h}$$

$$\overset{(GKS)}{\geq} 0$$

■

3.2 FKG INEQUALITY

We equip Ω with the product ordering

$$(\sigma \leq \sigma') \iff (\forall i \in \mathbb{Z}^d \quad \sigma_i \leq \sigma_i')$$

Def: A function $f : \Omega \to \mathbb{R}$ is <u>increasing</u> if $\forall \sigma, \sigma' \in \Omega$

$$\sigma \leq \sigma' \implies f(\sigma) \leq f(\sigma')$$

<u>Examples</u>: • $\sigma \mapsto \sigma_0$ is increasing.

• $\sigma \mapsto \sigma_A$ is <u>not</u> increasing when $|A| \geq 2$.

• $\forall i \in \mathbb{Z}^d$ $m_i : \sigma \mapsto \frac{\sigma_i + 1}{2}$ is increasing.

• $\forall A \subset\subset \mathbb{Z}^d$ $m_A := \prod_{i \in A} m_i$ is increasing.

<u>Thm (FKG inequality)</u>

Let $\Lambda \subset\subset \mathbb{Z}^d$, $\beta \geq 0$, $h \in \mathbb{R}$

• $\forall \omega \in \Omega$ $\forall f, g : \Omega_\Lambda^\omega \to \mathbb{R}$ increasing

$$\boxed{\langle f \times g \rangle^\omega_{\Lambda,\beta,h} \geq \langle f \rangle^\omega_{\Lambda,\beta,h} \langle g \rangle^\omega_{\Lambda,\beta,h}}$$

• $\forall f, g : \Omega_\Lambda \to \mathbb{R}$ increasing

$$\boxed{\langle f \times g \rangle^\emptyset_{\Lambda,\beta,h} \geq \langle f \rangle^\emptyset_{\Lambda,\beta,h} \langle g \rangle^\emptyset_{\Lambda,\beta,h}}$$

We will prove the FKG inequality later, on the chapter on stochastic domination

<u>Corollary</u>.

Let $\Lambda \subset\subset \mathbb{Z}^d$, $\beta \geq 0$, $h \in \mathbb{R}$.

1. [Monotonicity in h] $\forall \# \in \Omega$ or $\# = \emptyset$ $\forall h' \geq h$, we have

$\forall f$ increasing $\boxed{\langle f \rangle^\#_{\Lambda,\beta,h'} \geq \langle f \rangle^\#_{\Lambda,\beta,h}}$

2. [Monotonicity in the b.c.] $\forall \omega' \geq \omega$ we have

$\forall f$ increasing $\boxed{\langle f \rangle^{\omega'}_{\Lambda,\beta,h} \geq \langle f \rangle^{\omega}_{\Lambda,\beta,h}}$

Proof: Let $f : \Omega^{w}_{\Lambda} \to \mathbb{R}$ increasing.

We prove that for every $w' \geq w$ and $h' \geq h$

$$\langle f \rangle^{w'}_{\Lambda,\beta,h'} \geq \langle f \rangle^{w}_{\Lambda,\beta,h}$$

(the proof of (1) for $\# = \phi$ is similar)

Observe that for every $\sigma \in \Omega$

$$e^{-H^{w'}_{\Lambda,\beta,h'}(\sigma)} = g(\sigma) \times e^{-H^{w}_{\Lambda,\beta,h}(\sigma)}$$

where $g(\sigma) = \exp\Big(\beta \sum_{\substack{\{i,j\} \in \overline{E}_\Lambda \setminus E_\Lambda \\ i \in \Lambda}} \sigma_i (w'_j - w_j) + (h' - h) \sum_{i \in \Lambda} \sigma_i \Big)$

Since g is increasing the FKG inequality implies

$$\langle f \rangle^{w'}_{\Lambda,\beta,h'} = \frac{\langle f \times g \rangle^{w}_{\Lambda,\beta,h}}{\langle g \rangle^{w}_{\Lambda,\beta,h}} \underset{\text{FKG}}{\geq} \langle f \rangle^{w}_{\Lambda,\beta,h} \quad \blacksquare$$

Rk : (1) can also be proved using the derivative formula.

$$\frac{d}{dh} \langle f \rangle^{w}_{\Lambda,\beta,h} = \sum_{i \in \Lambda} \langle f \sigma_i \rangle^{w}_{\Lambda,\beta,h} - \langle f \rangle^{w}_{\Lambda,\beta,h} \langle \sigma_i \rangle^{w}_{\Lambda,\beta,h}$$

$$\underset{\text{FKG}}{\geq} 0 \quad \blacksquare$$

CHAPTER 2 :

— ISING MODEL IN INFINITE VOLUME

In this chapter, we define Ising measures on $\Omega = \{-1,1\}^{\mathbb{Z}^d}$ equiped with the product σ-algebra. Two approaches are possible :

- the first one is to take weak limits of finite volume measures. We will begin with this approach and define the measures $\mu^+_{\beta,h}$ or $\mu^-_{\beta,h}$ which emerge naturally as the respective weak limits of $\mu^+_{\Lambda_n,\beta,h}$ and $\mu^-_{\Lambda_n,\beta,h}$.
  The + and − boundary conditions are particularly "nice" in this context thanks to their monotonicity properties: if A is an increasing local event, if (Λ_n) is an increasing sequence of sets, then the sequences $(\mu^+_{\Lambda_n,\beta,h}(A))_n$ and $(\mu^-_{\Lambda_n,\beta,h}(A))$ are respectively increasing and decreasing, and therefore converge.
- the second approach is to use the Gibbs formalism. A measure on Ω is an infinite Ising measure (or Gibbs measure) if its marginals coincide with the finite volume measures when we condition to the configuration outside of a finite box. This second approach will be related to the first one, because the measures $\mu^-_{\beta,h}$ and $\mu^+_{\beta,h}$ will be important examples of Gibbs measure.

0. PRELIMINARIES

In order to construct infinite volume measures, we need an extension theorem from measure theory. Since we are working on the product space $\Omega = \{-1,1\}^{\mathbb{Z}^d}$, we use Kolmogorov extension's theorem (but other approaches can be used, e.g. Riesz theorem, see [1]). We briefly state Kolmogorov's theorem for the space Ω.

0.1 KOLMOGOROV EXTENSION THEOREM.

Notation: Let $\Lambda \subset\subset \mathbb{Z}^d$, and $\eta \in \Omega_\Lambda$. We define the cylinder.

$$\boxed{C_{\Lambda,\eta} = \{\sigma \in \Omega : \forall i \in S \ \sigma_i = \eta_i\}}$$

For fixed Λ, the cylinders form a finite partition of Ω, and generate a simple σ-algebra, denoted

$$\mathcal{F}_\Lambda := \sigma\left((C_{\Lambda,\eta})_{\eta \in \Omega_\Lambda} \right)$$

Then we write $\mathcal{F}$ for the σ-algebra generated by all the cylinders.

$$\mathcal{F} := \sigma\left((C_{\Lambda,\eta})_{\Lambda \subset\subset \mathbb{Z}^d, \eta \in \Omega_\Lambda} \right) = \sigma\left(\bigcup_{\Lambda \subset\subset \mathbb{Z}^d} \mathcal{F}_\Lambda \right)$$

Kolmogorov's theorem states that if we have defined "compatible" probability measures on $\mathcal{F}_\Lambda$ for every Λ then, there exists a unique extension to $\mathcal{F}$.

Thm: [Kolmogorov's extension theorem].

- Consider a function $\mu : \bigcup_{\Lambda \subset\subset \mathbb{Z}^d} \mathcal{F}_\Lambda \longrightarrow \mathbb{R}_+$ s.t.

  $\forall \Lambda \subset\subset \mathbb{Z}^d \quad \mu_{|\mathcal{F}_\Lambda}$ is a probability measure on $(\Omega_\Lambda, \mathcal{F}_\Lambda)$

Then there exists a unique probability measure $\bar{\mu}$ on $\mathcal{F}$ that coincides with μ on every $\mathcal{F}_\Lambda$, $\Lambda \subset\subset \mathbb{Z}^d$

Ref: Kolmogorov's theorem is presented in many probability books. The version above is taken from Villani's lecture notes (available at cedricvillani.org/for-mathematicians/lecture-notes [Section III.6.5], in French)

In this section, an important role will be played by local functions.

Def: A function $f : \Omega \to \mathbb{R}$ is said to be local if there exists $\Lambda \subset\subset \mathbb{Z}^d$ s.t. f is $\mathcal{F}_\Lambda$-measurable.

Rk: By definition, a function f is local if and only if there exists cylinders $C_1, \dots, C_k$ and $f_1, \dots, f_k \in \mathbb{R}$ s.t. $f = \sum_{i=1}^{k} f_i \mathbb{1}_{C_i}$.

Def. An event $E \in \mathcal{F}$ is said to be local if there exists $\Lambda \subset\subset \mathbb{Z}^d$ such that $E \in \mathcal{F}_\Lambda$.

Important remark

If a function $f : \Omega \to \mathbb{R}$ is $\mathcal{F}_\Lambda$-measurable for some $\Lambda \subset\subset \mathbb{Z}^d$, then $f = f((\sigma_i)_{i \in \Lambda})$.

Therefore f can be seen as a function

$$f : \Omega_\Lambda^\omega \longrightarrow \mathbb{R} \qquad (\text{for every } \omega \in \Omega)$$

or

$$f : \Omega_\Lambda \longrightarrow \mathbb{R}$$

In particular $\langle f \rangle^\omega_{\Lambda,\beta,h}$ and $\langle f \rangle^\emptyset_{\Lambda,\beta,h}$ are well defined.

The same remark as above applies for local events

Proposition:

The set of local functions is a vector-space generated by $(n_A)_{A \subset\subset \mathbb{Z}^d}$

Proof: By definition, a function is $\mathcal{F}_\Lambda$-measurable (for some $\Lambda \subset\subset \mathbb{Z}^d$) iff it is a linear combination of $(\mathbb{1}_{C_{\Lambda,\eta}})_{\eta \in \Omega_\Lambda}$. Therefore it suffices to show that for every $\Lambda \subset\subset \mathbb{Z}^d$ and every $\eta \in \Omega_\Lambda$,

$$\mathbb{1}_{C_{\Lambda,\eta}} \in \operatorname{Span}_{A \subset\subset \mathbb{Z}^d}(n_A).$$

This is easily proved by writing

$$\mathbb{1}_{C_{\Lambda,\eta}} = \prod_{i \in \Lambda : \eta_i = 1} n_i \times \prod_{i \in \Lambda : \eta_i = 0} (1 - n_i),$$

and expanding the product.

1 THE INFINITE-VOLUME MEASURES $\mu^+_{\beta,h}$ AND $\mu^-_{\beta,h}$

Notation: We write $\Lambda_n \uparrow \mathbb{Z}^d$ if for every n, $\Lambda_n \subset \Lambda_{n+1} \subset\subset \mathbb{Z}^d$ and

$$\mathbb{Z}^d = \bigcup_{n \geq 1} \Lambda_n$$

Thm: Let $\beta \geq 0$, $h \in \mathbb{R}$. There exist two probability measures $\mu^-_{\beta,h}$ and $\mu^+_{\beta,h}$ on $(\Omega, \mathcal{F})$ characterized by

$\forall E$ local event, $\forall \Lambda_n \uparrow \mathbb{Z}^d$
(i) $\mu^+_{\beta,h}(E) = \lim_{n\to\infty} \mu^+_{\Lambda_n,\beta,h}(E)$
(ii) $\mu^-_{\beta,h}(E) = \lim_{n\to\infty} \mu^-_{\Lambda_n,\beta,h}(E)$

Rk ① Since E is local $\mu^+_{\Lambda_n}(E)$ and $\mu^-_{\Lambda_n}(E)$ are well defined for n large enough. (see the important remark in Section 0.2)

② The definition of $\mu^+_{\beta,h}$ and $\mu_{\beta,h}$ does not depend on the chosen sequence $\Lambda_n \uparrow \mathbb{Z}^d$.

A key ingredient in the construction of $\mu^+_{\beta,h}$ and $\mu^-_{\beta,h}$ is the following monotonicity property.

Lemma: ("pushing boundary conditions")

Let $\beta \geq 0$, $h \in \mathbb{R}$. Let $\Delta \subset \Lambda \subset\subset \mathbb{Z}^d$.
For every increasing event $A \in \mathcal{F}_\Delta$, we have

$$\mu^+_{\Delta,\beta,h}(A) \geq \mu^+_{\Lambda,\beta,h}(A) \quad \text{and} \quad \mu^-_{\Delta,\beta,h}(A) \leq \mu^-_{\Lambda,\beta,h}(A)$$

In words, the lemma above states that the probability of an increasing events on Δ decreases when the + b.c. are pushed away from Δ, it increases when − b.c. are pushed away from Δ.

<u>Proof</u>: $\mu^+_{\Lambda,\beta,h}(A) = \mu^+_{\Lambda,\beta,h}(A \mid \forall i \in \Lambda \setminus \Delta \ \sigma_i = 1)$

$$= \frac{\mu^+_{\Lambda,\beta,h}(A \cap \{\forall i \in \Lambda \setminus \Delta \ \sigma_i = 1\})}{\mu^+_{\Lambda,\beta,h}(\{\forall i \in \Lambda \setminus \Delta \ \sigma_i = 1\})}$$

$$\overset{FKG'}{\geqslant} \mu^+_{\Lambda,\beta,h}(A) \qquad \blacksquare$$

<u>Proof of the theorem</u>

We only prove the existence of $\mu^+_{\beta,h}$ satisfying (i). The uniqueness is a consequence of Kolmogorov's theorem, and the proof for $\mu^-_{\beta,h}$ is the same.

<u>Step 1</u>: construction of $\mu^+_{\beta,h}$

Let $B_n = \{-n, \dots, n\}^d$. Let A be a <u>local increasing</u> event. By the lemma, the real sequence $(\mu^+_{B_n,\beta,h}(A))_{n \geq n_0}$ is decreasing (we choose n_0 large enough so that $\mu^+_{B_{n_0},\beta,h}(A)$ is well defined). Define

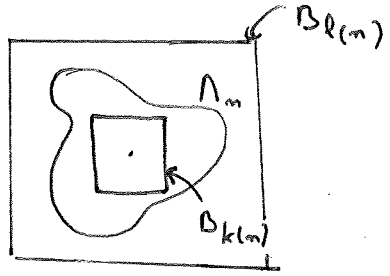
$$\mu^+_{\beta,h}(A) := \lim_{n\to\infty} \mu^+_{B_n,\beta,h}(A)$$

Now, let E be a <u>local event</u> (not necessarily increasing). By the proposition in section 0.2, $\mathbb{1}_E$ can be written a finite linear combination of $(\eta_A)_{A \subset\subset \mathbb{Z}^d}$. Therefore, $(\mu^+_{B_n,\beta,h}(E))_{n \geq n_0}$ converges, and we can define

$$\mu^+_{\beta,h}(E) := \lim_{n\to\infty} \mu^+_{B_n,\beta,h}(E)$$

We have defined $\mu^+_{\beta,h} : \bigcup_{\Lambda \subset\subset \mathbb{Z}^d} \mathcal{F}_\Lambda \to \mathbb{R}_+$ and $\forall \Lambda \subset\subset \mathbb{Z}$ $\mu^+_{\beta,h}|_{\mathcal{F}_\Lambda}$ is a probability measure on $\mathcal{F}_\Lambda$. By Kolmogorov's extension theorem, $\mu^+_{\beta,h}$ can be extended into a probability measure on $\mathcal{F}$.

<u>Step 2</u>: proof of (i). Let $\Lambda_m \uparrow \mathbb{Z}^d$. There exist two sequences $(k(m))_{m \geq 1}$ and $(l(m))_{m \in \mathbb{N}}$ s.t. $B_{k(m)} \subset \Lambda_m \subset B_{l(m)}$, and $k(m), l(m) \xrightarrow[m \to \infty]{} \infty$.



Using the lemma again, we find that for every local increasing event A,

$$\mu^+_{B_{k(m)},\beta,h}(A) \leq \mu^+_{\Lambda_m,\beta,h}(A) \leq \mu^+_{B_{l(m)},\beta,h}(A)$$

Hence, we also have $\lim_{m \to \infty} \mu^+_{\Lambda_m,\beta,h}(A) = \mu^+_{\beta,h}(A)$, which implies that the same convergence holds for every local event (as before, write $\mathbb{1}_E$ as a linear combination of $(n_A)_{A \subset\subset \mathbb{Z}^d}$)

■

<u>To Remember</u>:

$\mu^+_{\beta,h}$ is the "decreasing" limit of $(\mu^+_{\Lambda_m,\beta,h})$, in the sense that for every local increasing A,

$$\boxed{\mu^+_{\beta,h}(A) = \lim_{m \to \infty} \downarrow \mu^+_{\Lambda_m,\beta,h}(A)}$$

<u>Notation</u>: we write $\langle \cdot \rangle^-_{\beta,h}$ and $\langle \cdot \rangle^+_{\beta,}$ for the expectations with respect to $\mu^-_{\beta,h}$ and $\mu^+_{\beta,h}$.

<u>Def</u>: A measure μ on $(\Omega, \mathcal{F})$ is said to be <u>translation invariant</u> if for every $t \in \mathbb{Z}^d$,

$$\theta_t \# \mu = \theta_t$$

where $\theta_t : \Omega \to \Omega$ is defined by $\forall i \in \mathbb{Z}^d \ (\theta_t \sigma)_i = \theta_{i-t}$

Equivalently, μ is translation invariant iff for every $f:\Omega\to\mathbb{R}_+$

$$\int f\,d\mu = \int f\circ\theta_t^{-1}\,d\mu \qquad \forall t\in\mathbb{Z}^d.$$

Theorem:

> The measures $\mu^-_{\beta,h}$ and $\mu^+_{\beta,h}$ are translation invariant.

Pf: Since the local increasing events generate the σ-algebra $\mathcal{F}$, it suffices to prove that for every A local increasing,

$$\mu^+_{\beta,h}(\theta_t^{-1}(A)) = \mu^+_{\beta,h}(A).$$

First, let $\Lambda\subset\subset\mathbb{Z}^d$ and $\sigma\in\Omega^+_\Lambda$.

We have

$$H^+_{\Lambda-t,\beta,h}(\theta_t^{-1}(\sigma)) = -\beta\sum_{ij\in E_{\Lambda-t}}\sigma_{i+t}\sigma_{j+t} - h\sum_{i\in\Lambda-t}\sigma_{i+t}$$

$$= H^+_{\Lambda,\beta,h}(\sigma).$$

Therefore, for every $\sigma\in\Omega^+_\Lambda$

$$\mu^+_{\Lambda-t,\beta,h}(\theta_t^{-1}\sigma) = \mu^+_{\Lambda,\beta,h}(\sigma).$$

Now let $\Lambda_n\uparrow\mathbb{Z}^d$ and A local increasing. For every n large enough the equation above gives.

$$\mu^+_{\Lambda_n-t,\beta,h}(\theta_t^{-1}A) = \mu^+_{\Lambda_n,\beta,h}(A).$$

Since $(\Lambda_n-t)\uparrow\mathbb{Z}^d$, we obtain the result by taking the limit as n tends to infinity. ∎

2 GENERAL INFINITE-VOLUME ISING MEASURES

Not. : Write for $S \subset \mathbb{Z}^d$ (not necessarily finite), $\mathcal{F}_S$ for the σ-algebra generated by $(\mathcal{F}_\Lambda)_{\Lambda \subset\subset S}$.

Definition:

A measure on $(\Omega, \mathcal{F})$ is called an infinite-volume Ising measure (or Gibbs measure) at inverse temperature β, external field h if for every $\Lambda \subset\subset \mathbb{Z}^d$, and every f $\mathcal{F}_\Lambda$-meas.,

$$\boxed{\mu\left[f \mid \mathcal{F}_{\Lambda^c}\right](\sigma) = \langle f \rangle^{\sigma}_{\Lambda,\beta,h} \quad \text{for a.e. } \sigma \in \Omega.}$$

Proposition:

The two measures $\mu^+_{\beta,h}$ and $\mu^-_{\beta,h}$ are infinite-volume Ising measures at (β, h).

Proof:

Let $\Lambda \subset\subset \mathbb{Z}^d$. Let f be a $\mathcal{F}_\Lambda$-measurable function. We need to proove that for every $E \in \mathcal{F}_{\Lambda^c}$,

$$\langle f(\sigma)\, \mathbb{1}_E(\sigma) \rangle^+_{\beta,h} = \langle \langle f \rangle^{\sigma}_{\Lambda,\beta,h}\, \mathbb{1}_E(\sigma) \rangle^+_{\beta,h}$$

Since an event $E \in \mathcal{F}_{\Lambda^c}$ can be approximated by local events $E_1, \dots, E_n, \dots \in \mathcal{F}_{\Lambda^c}$ satisfying $\mu^+_{\beta,h}[E_n \Delta E] \underset{n\to\infty}{\to} 0$, it suffices to prove the equation above for $E \in \mathcal{F}_{\Lambda^c}$ local. Fix $E \in \mathcal{F}_{\Lambda^c}$ local. Let $\Lambda_n \uparrow \mathbb{Z}^d$. Let $n_0 \geq 1$ large enough so that $\Lambda \subset \Lambda_{n_0}$ and E is $\mathcal{F}_{\Lambda_{n_0}}$-measurable. Recall that for every $\omega \in \Omega^+_{\Lambda_n}$,

$$\langle f \mid \forall i \in \Lambda_n \setminus \Lambda \ \sigma_i = \omega_i \rangle^+_{\Lambda_n,\beta,h} = \langle f \rangle^{\omega}_{\Lambda,\beta,h},$$

In other words, if we consider the σ-algebra $\mathcal{G}$ in $\Omega^+_{\Lambda_m}$ generated by $(\omega \mapsto \sigma_i)_{i \in \Lambda_m \setminus \Lambda}$, we have $\forall \sigma \in \Omega^+_{\Lambda_m,\beta,h}$

$$\langle f \mid \mathcal{G} \rangle^+_{\Lambda_m,\beta,h}(\sigma) = \langle f \rangle^\sigma_{\Lambda,\beta,h}$$

For $m \geq m_0$, the event E can be seen as an element of $\mathcal{G}$, and therefore,

$$\langle f(\sigma)\, 1\!\!1_E(\sigma) \rangle^+_{\Lambda_m,\beta,h} = \langle \langle f \mid \mathcal{G} \rangle^+_{\Lambda_m,\beta,h}(\sigma)\, 1\!\!1_E(\sigma) \rangle^+_{\Lambda_m,\beta,h}$$

$$= \langle \langle f \rangle^\sigma_{\Lambda,\beta,h}\, 1\!\!1_E(\sigma) \rangle^+_{\Lambda_m,\beta,h}.$$

Now, observe that $\sigma \to \langle f \rangle^\sigma_{\Lambda,\beta,h}$ is local because the measure $\mu^\sigma_{\Lambda,\beta,h}$ depends on σ only through the values of σ at the boundary of Λ. Hence we can take the limit as m tends to infinity in the equation above, which concludes the proof ∎

The following proposition shows that the measures $\mu^+_{\beta,h}$ and $\mu^-_{\beta,h}$ play specific roles: they are extremal for the stochastic ordering.

Prop. Let μ be an infinite-volume Ising measure. Then, for every local increasing A,

$$\boxed{\mu^-_{\beta,h}(A) \leq \mu(A) \leq \mu^+_{\beta,h}(A).}$$

Proof: Let $\Lambda_m \uparrow \mathbb{Z}^d$. For m large enough, and for every $\omega \in \Omega$

$$\mu^-_{\Lambda_m,\beta,h}(A) \leq \mu^\omega_{\Lambda_m,\beta,h}(A) \leq \mu^+_{\Lambda_m,\beta,h}(A),$$

Since μ is an infinite-volume Ising measure, it gives,

$$\mu^-_{\Lambda;\beta,h}(A) \leq \mu(A \mid \mathcal{F}_{\Lambda^c}) \leq \mu^+_{\Lambda;\beta,h}(A)$$

Taking the expectation with respect to μ, and letting n tend to infinity gives the desired inequalities. ■

We denote by $\mathcal{G}(\beta, h)$ the set of all infinite-volume Ising measures at $\beta \geq 0$ $h \in \mathbb{R}$.

Theorem: (First characterization of uniqueness).

Let $\beta \geq 0$ $h \in \mathbb{R}$. The following are equivalent

(i) There exists a unique infinite volume Ising measure at (β, h).

(ii) $\mu^-_{\beta,h} = \mu^+_{\beta,h}$

(iii) $\langle \sigma_0 \rangle^-_{\beta,h} = \langle \sigma_0 \rangle^+_{\beta,h}$

Proof:

(i)$\Rightarrow$(ii) follows directly from the fact that $\mu^-_{\beta,h} = \mu^+_{\beta,h}$

(ii)$\Rightarrow$(i) If $\mu^-_{\beta,h} = \mu^+_{\beta,h}$, the proposition above implies that for every $\mu \in \mathcal{G}(\beta,h)$, and every local increasing A,

$$\mu^-_{\beta,h}(A) = \mu(A) = \mu^+_{\beta,h}(A).$$

Since the local increasing events generate the σ-algebra $\mathcal{F}$, this implies that $\mu = \mu^-_{\beta,h} = \mu^+_{\beta,h}$.

(ii)$\Rightarrow$(iii) trivial

(iii)$\Rightarrow$(ii) Fix $A \subset\subset \mathbb{Z}^d$. Since the function $\sum_{i \in A} n_i - n_A$ is increasing, we have

$$\Big\langle \sum_{i \in A} n_i - n_A \Big\rangle^-_{\beta,h} \leq \Big\langle \sum_{i \in A} n_i - n_A \Big\rangle^+_{\beta,h}$$

Therefore,

$$\langle n_A \rangle^+_{\beta,h} - \langle n_A \rangle^-_{\beta,h} \leq \sum_{i \in A} \langle n_i \rangle^+_{\beta,h} - \langle n_i \rangle^-_{\beta,h}$$

$$= \sum_{i \in A} \langle \sigma_i \rangle^+_{\beta,h} - \langle \sigma_i \rangle^-_{\beta,h}$$

$$\underset{\substack{\uparrow \\ \text{translation invariance}}}{=} |A| \left(\langle \sigma_0 \rangle^+_{\beta,h} - \langle \sigma_0 \rangle^-_{\beta,h} \right) \underset{\substack{\uparrow \\ \text{(iii)}}}{=} 0$$

Since the functions $(m_A)_{A\subset\subset\mathbb{Z}^d}$ generate the σ-algebra $\mathcal{F}$, we deduce that the measures $\mu^-_{\beta,h}$ and $\mu^+_{\beta,h}$ are equal. ■

3) UNIQUENESS VIA CONVEXITY OF THE PRESSURE.

In this section, we will provide some analytic characterizations of uniqueness, using convexity and continuity arguments. First, we give some simple properties of the magnetization $\langle\sigma_0\rangle^+_{\beta,h}$ (and $\langle\sigma_0\rangle^-_{\beta,h}$) which follow from the fact that it is a monotone limit of the continuous increasing functions (in h) $\langle\sigma_0\rangle^+_{\Lambda_n,\beta,h}$ (and $\langle\sigma_0\rangle^-_{\Lambda_n,\beta,h}$).

Prop.: [properties of the magnetization]

Fix $\beta\geq 0$.

(i) The function $h\mapsto\langle\sigma_0\rangle^+_{\beta,h}$ is increasing, right-continuous.

(ii) The function $h\mapsto\langle\sigma_0\rangle^-_{\beta,h}$ is increasing, left-continuous.

(iii) For $B_n=\{-n,\dots,n\}^d$, we have

$$\langle\sigma_0\rangle^+_{\beta,h}=\lim_{n\to\infty}\frac{1}{|B_n|}\sum_{i\in B_n}\langle\sigma_i\rangle^+_{\beta,h},$$

$$\langle\sigma_0\rangle^-_{\beta,h}=\lim_{n\to\infty}\frac{1}{|B_n|}\sum_{i\in B_n}\langle\sigma_i\rangle^-_{\beta,h}.$$

Proof: (i)(ii) follow from the facts that $h\mapsto\langle\sigma_0\rangle^+_{\Lambda_n,\beta,h}$ and $h\mapsto\langle\sigma_0\rangle^-_{\Lambda_n,\beta,h}$ are continuous increasing and

$$\langle\sigma_0\rangle^+_{\beta,h}=\inf_{n\in\mathbb{N}}\langle\sigma_0\rangle^+_{\Lambda_n,\beta,h}\quad\text{and}\quad\langle\sigma_0\rangle^-_{\beta,h}=\sup_{n\in\mathbb{N}}\langle\sigma_0\rangle^-_{\Lambda_n,\beta,h}$$

(iii) First, for every $N \geq m$, we have

$$\forall i \in \Lambda_m \quad \langle \sigma_i \rangle^+_{B_m,\beta,h} \geq \langle \sigma_i \rangle^+_{B_N,\beta,h}$$

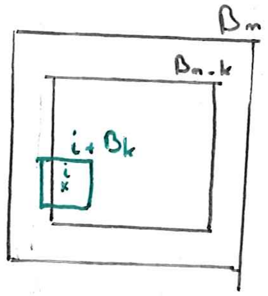
Taking the limit as N tends to infinity, an using translation invariance, we obtain $\langle \sigma_i \rangle^+_{B_m,\beta,h} \geq \langle \sigma_0 \rangle^+_{\beta,h}$, which directly implies

$$\frac{1}{|B_m|} \sum_{i \in B_m} \langle \sigma_i \rangle^+_{B_m} \geq \langle \sigma_0 \rangle^+_{\beta,h} .$$

Now, fix $k \leq m$. We have

$$\forall i \in B_{m-k} \quad \langle \sigma_i \rangle^+_{B_m,\beta,h} \leq \underbrace{\langle \sigma_i \rangle^+_{i+B_k,\beta,h}}_{= \langle \sigma_0 \rangle^+_{B_k,\beta,h}}$$

since $i + B_k \subset B_{m-k}$.



using the trivial bound $\langle \sigma_i \rangle^+_{B_m,\beta,h} \leq 1$ for $i \in B_m \setminus B_{m-k}$, we obtain

$$\frac{1}{|B_m|} \sum_{i \in B_m} \langle \sigma_i \rangle^+_{B_m,\beta,h} \leq \underbrace{\frac{|B_{m-k}|}{|B_m|}}_{\xrightarrow[m\to\infty]{} 1} \langle \sigma_0 \rangle^+_{B_k,\beta,h} + \underbrace{\frac{|B_m \setminus B_{m-k}|}{B_m}}_{\xrightarrow[m\to\infty]{} 0}$$

Therefore,

$$\limsup_{m\to\infty} \frac{1}{|B_m|} \sum_{i \in B_m} \langle \sigma_i \rangle^+_{B_m,\beta,h} \leq \langle \sigma_0 \rangle^+_{B_k,\beta,h} .$$

We finally obtain the result by letting k tend to infinity.

Notation: For $\Lambda \subset\subset \mathbb{Z}^d$, for $\# \in \{\emptyset\} \cup \Omega$,

$$\boxed{p^{\#}_{\Lambda}(\beta, h) = \frac{1}{|\Lambda|} \log Z^{\#}_{\Lambda,\beta,h}}$$

Thm [Definition of the pressure]

Let $B_n = \{-2^n, \dots, 2^n - 1\}$. Let $\beta \geq 0$, $h \in \mathbb{R}$.
For every b.c. $\# \in \{\emptyset\} \cup \Omega$, the sequence $\left(p^{\#}_{B_n}(\beta,h)\right)_n$ converges and the limit

$$p(\beta, h) := \lim_{n\to\infty} p^{\#}_{B_n}(\beta,h)$$

does not depend on the b.c. $\#$.

The quantity $p(\beta,h)$ is called the pressure.

Remark: The choice of B_n is important here in the sense that

$$\frac{|\partial B_n|}{|B_n|} \xrightarrow[n\to\infty]{} 0 .$$

Exercise: (not easy)

- Give an example of $\Lambda_n \uparrow \mathbb{Z}^d$, $\beta \geq 0$, $h \in \mathbb{R}$, $\# \in \{\emptyset\} \cup \Omega$, for which $p^{\#}_{\Lambda_n}(\beta,h)$ does not converge to $p(\beta,h)$
- Prove that for every $\# \in \{\emptyset\} \cup \Omega$,

$$\forall \Lambda_n \uparrow \mathbb{Z}^d \text{ s.t. } \frac{|\partial \Lambda_n|}{|\Lambda_n|} \xrightarrow[n\to\infty]{} 0, \quad \lim_{n\to\infty} p^{\#}_{\Lambda_n}(\beta,h) = p(\beta,h)$$

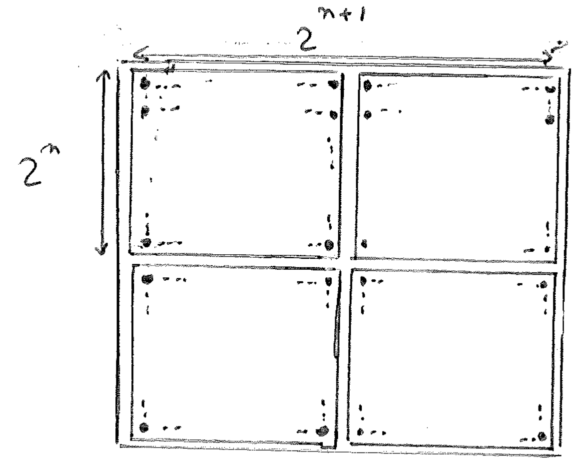
where $\partial \Lambda_n = \{ i \in \mathbb{Z}^d \setminus \Lambda_n \text{ s.t. } \exists j \in \Lambda_n \ \|i-j\|_1 = 1 \}$

Proof of thm:

We begin with the free b.c. $\# = \emptyset$.

For $n \geq 1$ we consider a covering of B_{n+1} by 2^d boxes of radius 2^n. We consider $B^{(1)}, \dots, B^{(2^d)}$ such that

$$B_{n+1} = B^{(1)} \cup \dots \cup B^{(2^d)} \quad \text{and} \quad \begin{cases} B_i^{(i)} \cap B^{(j)} = \emptyset & \text{for } i \neq j \\ B^{(i)} \text{ is a translate of } B_n. \end{cases}$$



For $\sigma \in \Omega_{B_n}$ we write $\sigma^{(k)} = \sigma_{|B^{(k)}}$

Then the energy of σ can be decomposed as.

$$H^{\emptyset}_{B_n,\beta,h}(\sigma) = \sum_{k=1}^{2^d} H^{\emptyset}_{B^{(k)},\beta,h}(\sigma^{(k)}) \underbrace{- \beta \sum_{\{i,j\} \in E_{B_n} \setminus \cup E_{B^{(k)}}} \sigma_i \sigma_j}_{\geq -d \times 2^{(d-1)(n+1)}}$$

Hence,

$$Z^{\emptyset}_{B_{n+1},\beta,h} = \sum_{\sigma \in \Omega_{B_n}} e^{-H^{\emptyset}_{B_n,\beta,h}(\sigma)}$$

$$\leq \sum_{\sigma^{(1)} \in \Omega_{B^{(1)}}} \cdots \sum_{\sigma^{(2^d)} \in \Omega_{B^{(d)}}} e^{-H^{\emptyset}_{B^{(1)},\beta,h}(\sigma^{(1)})} \times \dots \times e^{-H_{B^{(2^d)},\beta,h}(\sigma^{(d)})} \times e^{d \times 2^{(d-1)(n+1)}}$$

$$= Z^{\emptyset}_{B^{(1)},\beta,h} \times \dots \times Z^{\emptyset}_{B^{(2^d)},\beta,h} \times e^{d\, 2^{(d-1)(n+1)}}$$

$$= \left(Z^{\phi}_{B_n,\beta,h}\right)^{2^d} \times e^{d\, 2^{(d-1)(n+1)}}.$$

We obtain

$$f^{\emptyset}_{B_{n+1}}(\beta,h) = \frac{1}{|B_{n+1}|} \log\left(Z^{\emptyset}_{B_{n+1},\beta,h}\right)$$

$$\leq \frac{2^d \log(Z^{\emptyset}_{B_n,\beta,h})}{2^d \cdot |B_n|} + \frac{\beta d \times 2^{(n+1)(d-1)}}{2^{(n+1)d}}$$

$$= f^{\emptyset}_{B_n}(\beta,h) + \beta d \cdot \frac{1}{2^{n+1}}$$

Equivalentely we can prove that $f^{\emptyset}_{B_{n+1}}(\beta) \geq f^{\emptyset}_{B_n}(\beta,h) - \beta d \cdot \frac{1}{2^{n+1}}$

This implies that $\left(f^{\emptyset}_{B_n}(\beta,h)\right)_n$ is a Cauchy sequence, and therefore converges to a limit $f(\beta,h)$.

Now, let $\omega \in \Omega$ be arbitrary b.c. For every $\sigma \in \Omega^{\omega}_{B_n}$, we have $H^{\omega}_{B_n,\beta,h}(\sigma) = H^{\emptyset}_{B_n,\beta,h}(\sigma) - \beta \sum_{\{i,j\} \in \overline{E}_{B_n} \setminus E_{B_n}} \sigma_i \sigma_j$

Hence $\left| H^{\omega}_{B_n,\beta,h}(\sigma) - H^{\emptyset}_{B_n,\beta,h}(\sigma) \right| \leq \beta \, |\partial B_n|$,

therefore $\dfrac{\log(Z^{\omega}_{B_n,\beta,h}) - \log(Z^{\emptyset}_{B_n,\beta,h})}{|B_n|} \leq \beta \dfrac{|\partial B_n|}{|B_n|}$.

which implies $\lim_{n\to\infty} \left| f^{\omega}_{B_n}(\beta,h) - f(\beta,h) \right| = 0$ ∎

Thm [Analytic properties of the pressure]

- The function $f : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is convex.
- For every $\beta \geq 0$ and $h \in \mathbb{R}$, we have

$$\frac{\partial f}{\partial h^+}(\beta,h) = \langle \sigma_0 \rangle^+_{\beta,h} \quad \text{and} \quad \frac{\partial f}{\partial h^-}(\beta,h) = \langle \sigma_0 \rangle^-_{\beta,h}.$$

Pf: Since a pointwise limit of convex functions is convex, it suffices to prove that for every $m \in \mathbb{N}$ $f^{\emptyset}_{B_m}$ is convex.

Let $\lambda = \sum_{\sigma \in \Omega_{B_m}} \delta_\sigma$ be the counting measure on Ω_{B_m}.

$$X(\sigma) := \Big(\sum_{\{i,j\} \in E_{B_m}} \sigma_i \times \sigma_j \;,\; \sum_{i \in B_m} \sigma_i \Big)$$

We can write $\forall (\beta, h) \in \mathbb{R}_+ \times \mathbb{R}$

$$Z^{\emptyset}_{B_m, \beta, h} = \int_{\Omega_{B_m}} e^{(\beta,h)\cdot X} \, d\lambda$$

Now, let $(\beta, h), (\beta', h') \in \mathbb{R}_+ \times \mathbb{R}$ $\alpha \in [0,1]$

$$Z^{\emptyset}_{B_m, \alpha\beta + (1-\alpha)\beta', \alpha h + (1-\alpha) h'} = \int_{\Omega_{B_m}} e^{\alpha(\beta,h)\cdot X} \times e^{(1-\alpha)(\beta',h')\cdot X} \, d\lambda$$

$$\leq \Big(\int e^{(\beta,h)\cdot X} \Big)^{\alpha} \cdot \Big(\int e^{(\beta',h')\cdot X} \Big)^{1-\alpha}$$

Hölder with
$p = \frac{1}{\alpha}$ $q = \frac{1}{1-\alpha}$

$$= \Big(Z^{\emptyset}_{B_m, \beta, h} \Big)^{\alpha} \times \Big(Z^{\emptyset}_{B_m, \beta', h'} \Big)^{1-\alpha}.$$

Taking the logarithm and dividing by $|B_m|$, we finally get

$$f^{\emptyset}_{B_m}\big(\alpha(\beta,h) + (1-\alpha)(\beta',h')\big) \leq \alpha f^{\emptyset}_{B_m}(\beta,h) + (1-\alpha) f^{\emptyset}_{B_m}(\beta',h')$$

which concludes that f is convex.

It remains to compute the left and right derivatives of f.

First, fix $m \in \mathbb{N}$. For $(\beta, h) \in \mathbb{R}_+ \times \mathbb{R}$, we have

$$\frac{\partial}{\partial h}\left(Z^+_{B_m,\beta,h} \right) = \frac{\partial}{\partial h}\left(\sum_{\sigma \in \Omega^+_{B_m}} e^{\beta \sum_{\{i,j\} \in E_{B_m}} \sigma_i \sigma_j + h \sum_{i \in B_m} \sigma_i} \right)$$

$$= \sum_{\sigma \in \Omega^+_{B_m}} \left(\sum_{i \in B_m} \sigma_i \right) e^{-H^+_{B_m,\beta,h}(\sigma)}$$

Therefore

$$\frac{\partial}{\partial h} f^+_{B_m}(\beta, h) = \frac{\partial}{\partial h}\left(\frac{1}{|B_m|} \log\left(Z^+_{B_m,\beta,h} \right) \right)$$

$$= \frac{1}{|B_m|} \left\langle \sum_{i \in B_m} \sigma_i \right\rangle^+_{B_m,\beta,h}$$

Now fix $(\beta_0, h_0) \in \mathbb{R}_+ \times \mathbb{R}$. Using that $\langle \sigma_i \rangle^+_{B_m,\beta,h}$ is increasing in h, the mean-value theorem implies for every $h > h_0$

$$\frac{1}{|B_m|} \sum_{i \in B_m} \langle \sigma_i \rangle^+_{B_m,\beta_0,h_0} \leq \frac{f^+_{B_m}(\beta_0, h) - f^+_{B_m}(\beta_0, h_0)}{h - h_0} \leq \frac{1}{|B_m|} \sum_{i \in B_m} \langle \sigma_i \rangle^+_{B_m,\beta_0,h}$$

Taking the limit as m tends to infinity gives

$$\langle \sigma_0 \rangle^+_{\beta_0,h_0} \leq \frac{f(\beta_0, h) - f(\beta_0, h_0)}{h - h_0} \leq \langle \sigma_0 \rangle^+_{\beta_0,h}$$

Using that $\langle \sigma_0 \rangle^+_{\beta_0,h}$ is a right-continuous function of h, we finally get that $\lim_{h \searrow h_0} \frac{f(\beta_0, h) - f(\beta_0, h_0)}{h - h_0} = \langle \sigma_0 \rangle^+_{\beta_0,h_0}$.

The left derivative is computed equivalently, using that $h \to \langle \sigma_0 \rangle^-_{\beta_0,h}$ is left-continuous.

This theorem implies directly that there is a unique infinite volume Ising measure at (β,h) if and only if $f(\beta,\cdot)$ is differentiable at h. Furthermore, by convexity, we know that for fixed β, the function $f(\beta,\cdot)$ is differentiable at every h except possibly at countably many points. Therefore, an important consequence of the theorem above is that for fixed β, the set of h for which there is no uniqueness at (β,h) is at most countable.
This leads to the following theorem:

<u>Thm</u> [Second characterization of uniqueness]

Fix $(\beta,h)\in\mathbb{R}_+\times\mathbb{R}$. The following are equivalent

(i) There exists a unique infinite volume Ising measure at (β,h)

(ii) $f(\beta,\cdot)$ is differentiable at h.

(iii) $h\mapsto\langle\sigma_0\rangle^+_{\beta,h}$ is continuous at h.

(iv) $h\mapsto\langle\sigma_0\rangle^-_{\beta,h}$ is continuous at h.

<u>Pf</u>: (i)$\Leftrightarrow$(ii) is a trivial consequence of the previous theorem.

(i)$\Leftrightarrow$(iii) Since there exist at most countably many points where $f(\beta,\cdot)$ is not differentiable, we can pick $h_n\uparrow h$ s.t. $\forall n$ $\langle\sigma_0\rangle^+_{\beta,h_n}=\langle\sigma_0\rangle^-_{\beta,h_n}$
Since $\langle\sigma_0\rangle^+_{\beta,h}$ is a right-continuous increasing function of h, we have

$$\left(h\mapsto\langle\sigma_0\rangle^+_{\beta,h}\text{ continuous at }h\right)\Leftrightarrow\left(\langle\sigma_0\rangle^+_{\beta,h}=\lim_{n\to\infty}\langle\sigma_0\rangle^+_{\beta,h_n}\right)$$

By our choice of (h_n) we have

$$\lim_{n\to\infty} \langle\sigma_0\rangle^+_{\beta,h_n} = \lim_{n\to\infty} \langle\sigma_0\rangle^-_{\beta,h_n}$$

$$= \langle\sigma_0\rangle^-_{\beta,h}$$

(left-continuity of $\langle\sigma_0\rangle^-_{\beta,h}$ in h)

which concludes that (iii) is equivalent to $\langle\sigma_0\rangle^+_{\beta,h} = \langle\sigma_0\rangle^-_{\beta,h}$

The equivalence (iv) $\Leftrightarrow$ (i) is proved in the same way

4) Uniqueness in non-zero magnetic field.

$\boxed{\text{Thm: For every } \beta \geq 0 \text{ and every } h \in \mathbb{R}\setminus\{0\}, \text{ there exists a unique infinite-volume Ising measure.}}$

Remark: We will prove the theorem above by showing that for every fixed $\beta \geq 0$, $h \to \langle\sigma_0\rangle^+_{\beta,h}$ is continuous on $\mathbb{R}\setminus\{0\}$. This is equivalent to the fact that $h \to f(\beta,h)$ is differentiable. Actually, the pressure is very regular on $\mathbb{R}_+ \times (\mathbb{R}\setminus\{0\})$: Lee-Yang Theorem states that f is analytic on $\mathbb{R}_+ \times (\mathbb{R}\setminus\{0\})$.

We will rely on the GHS inequality (Griffiths, Hurst, Sherman) that will be proved later in the random currents' chapter.

Thm: (GHS inequality)

Let $\Lambda \subset\subset \mathbb{Z}^d$. Let $\beta \geq 0$, $\underline{h \geq 0}$. We have for every $i,j,k \in \Lambda$

$$\boxed{\langle\sigma_i;\sigma_j;\sigma_k\rangle^+_{\Lambda,\beta,h} \leq 0}$$

where $\langle\sigma_i;\sigma_j;\sigma_k\rangle^+_{\Lambda,\beta,h} := \langle\sigma_i\sigma_j\sigma_k\rangle^+_{\Lambda,\beta,h} - \langle\sigma_i\rangle^+_{\Lambda,\beta,h}\langle\sigma_j\sigma_k\rangle^+_{\Lambda,\beta,h} - \langle\sigma_j\rangle^+_{\Lambda,\beta,h}\langle\sigma_i\sigma_k\rangle^+_{\Lambda,\beta,h}$
$- \langle\sigma_k\rangle^+_{\Lambda,\beta,h}\langle\sigma_i\sigma_j\rangle^+_{\Lambda,\beta,h} + 2\langle\sigma_i\rangle^+_{\Lambda,\beta,h}\langle\sigma_j\rangle^+_{\Lambda,\beta,h}\langle\sigma_k\rangle^+_{\Lambda,\beta,h}$

Rk: The GHS inequality is also true if we replace the + b.c. with free b.c. $\emptyset$, but the fact that $h \geq 0$ is important, and we cannot take − b.c.

Proof of the first theorem.

As in the proof of the monotonicity of $\langle \sigma_0 \rangle^+_{\Lambda,\beta,h}$ in β, we can prove that for every $A \subset \Lambda \subset\subset \mathbb{Z}^d$

$$\frac{d}{dh} \langle \sigma_A \rangle^+_{\Lambda,\beta,h} = \sum_{x \in \Lambda} \langle \sigma_A \sigma_x \rangle^+_{\Lambda,\beta,h} - \langle \sigma_A \rangle^+_{\Lambda,\beta,h} \langle \sigma_x \rangle^+_{\Lambda,\beta,h}$$

Therefore, if we consider $g(h) := \langle \sigma_0 \rangle^+_{\Lambda,\beta,h}$

we see that $$g'(h) = \sum_{x \in \Lambda} \langle \sigma_0 \sigma_x \rangle^+_{\Lambda,\beta,h} - \langle \sigma_0 \rangle^+_{\Lambda,\beta,h} \langle \sigma_x \rangle^+_{\Lambda,\beta,h}$$

and $$g''(h) = \sum_{x,y \in \Lambda} \langle \sigma_0 ; \sigma_x ; \sigma_y \rangle^+_{\Lambda,\beta,h}$$

Hence, the GHS inequality implies that for every $\Lambda \subset\subset \mathbb{Z}^d$ and every fixed $\beta \geq 0$, $h \mapsto \langle \sigma_0 \rangle^+_{\Lambda,\beta,h}$ is concave on $\mathbb{R}_+$.

Now by considering a sequence $\Lambda_n \uparrow \mathbb{Z}^d$, and using tha a simple limit of concave function is concave, we obtain that

$$h \mapsto \langle \sigma_0 \rangle^+_{\beta,h} \text{ is concave on } \mathbb{R}_+$$

Therefore, it is continuous on $(0, \infty)$. Using the second characterisation of uniqueness, this implies that there is a unique infinite-volume Ising measure for every $h > 0$.

For $h \leq 0$ we have $\langle \sigma_0 \rangle^-_{\beta,h} = -\langle \sigma_0 \rangle^+_{\beta,-h}$ (by passing the relation $\langle \sigma_0 \rangle^-_{\Lambda_n,\beta,h} = -\langle \sigma_0 \rangle^+_{\Lambda_n,\beta,-h}$ to the limit).

This implies that $h \mapsto \langle \sigma_0 \rangle^-_{\beta,h}$ is convex on $\mathbb{R}^-$, continuous on $(-\infty, 0)$, which concludes that there exists a unique infinite-volume Ising measure when $h < 0$ ∎

CHAPTER 3 : PHASE TRANSITION OF THE ISING MODEL —

1) INTRODUCTION, DEFINITION OF β_c.

In the previous chapter, we have seen that there is always a unique infinite volume Ising measure when the external field satisfies $h \neq 0$. Therefore, non-uniqueness may occur only on the half-line $(h=0, \beta \geq 0)$.

In all this chapter we work with $h=0$, and drop the dependence on h from our notation, and write $\mu^{\#}_{\Lambda,\beta}$, $\langle \cdot \rangle^{\#}_{\Lambda,\beta}$, $\mu^{\#}_{\beta}$, $\langle \cdot \rangle^{\#}_{\beta}$ in place of $\mu^{\#}_{\Lambda,\beta,0}$, $\langle \cdot \rangle^{\#}_{\Lambda,\beta,0}$; $\mu^{\#}_{\beta,0}$, $\langle \cdot \rangle^{\#}_{\beta,0}$ for every admissible bondary conditions.

An important property of the model when $h=0$ is the spin-flip symmetry. In particular, it implies $\langle \sigma_0 \rangle^{-}_{\beta} = -\langle \sigma_0 \rangle^{+}_{\beta}$ for every $\beta \geq 0$. Hence the condition $\langle \sigma_0 \rangle^{-}_{\beta} = \langle \sigma_0 \rangle^{+}_{\beta}$ is equivalent to $\langle \sigma_0 \rangle^{+}_{\beta} = 0$. Therefore, when $h=0$, the first characterization of uniqueness can be rewritten as follows

Thm (characterization of uniqueness when $h=0$)

The following are equivalent:

(i) there exists a unique infinite-volume Ising measure.

(ii) $\mu^{-}_{\beta} = \mu^{+}_{\beta}$

(iii) $\langle \sigma_0 \rangle^{+}_{\beta} = 0$

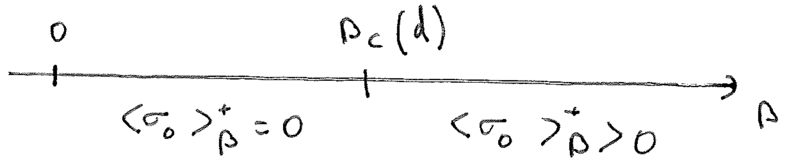
Phase transition of Ising model.

Recall that $\beta \mapsto \langle \sigma_0 \rangle_\beta^+$ is increasing in β.

Define

$$\boxed{\beta_c(d) := \sup\{\beta \geq 0 : \langle \sigma_0 \rangle_\beta^+ = 0\}}$$

By monotonicity, we have $\langle \sigma_0 \rangle_\beta^+ \begin{cases} = 0 & \forall \beta < \beta_c(d) \\ > 0 & \forall \beta > \beta_c(d) \end{cases}$



Exercise: Check that $\langle \sigma_0 \rangle_0^+ = 0$.

The goal of this chapter is to answer the following non-trivial question:

Is the critical value $\beta_c(d)$ non-trivial?

Or, in other words, do we have $\beta_c(d) \in (0, \infty)$?
In his thesis, Ising proved that there is no phase transition in dimension $d = 1$ (he proved $\beta_c(1) = \infty$).

The goal in this chapter is to show that in any dimension $d \geq 2$ we have $\beta_c(d) \in (0, \infty)$. To do this, we will use different representations of the Ising model:

- Using the high temperature expansion, we will prove that $\beta_c > 0$.
- Using the low temperature expansion, we will prove $\beta_c < \infty$, relying on the Peierls's argument.

The general idea in these representations is to rewrite the quantity

$$\langle \sigma_0 \rangle^+_{\Lambda,\beta} = \frac{\sum_{\sigma \in \Omega^+_\Lambda} \sigma_0 \, e^{-H^+_\beta(\sigma)}}{\sum_{\sigma \in \Omega^+_\Lambda} e^{-H^+_\beta(\sigma)}}$$

in terms of sums over different combinatorial objects.

More precisely, we define a class of objects $\mathcal{X}$ (in the low temperature expansion, $\mathcal{X}$ are sets of contours, in the high temperature expansion, $\mathcal{X}$ are subgraphs of Λ)

such that

$$\langle \sigma_0 \rangle^+_{\Lambda,\beta} = \frac{\sum_{x \in \mathcal{X}_1} f(x) \, w(x)}{\sum_{x \in \mathcal{X}_0} w(x)}$$

where $\begin{cases} \mathcal{X}_0, \mathcal{X}_1 \subset \mathcal{X} \\ f : \mathcal{X} \to \mathbb{R}. \\ w : \mathcal{X} \to \mathbb{R} \text{ gives to each element } x \text{ a weight } w(x). \end{cases}$

The general idea to obtain such a represetation is to first rewrite the partition function as

$$Z^+_{\Lambda,\beta} = \sum_{x \in \mathcal{X}_0} w(x)$$

and then try to write

$$\sum_{\sigma \in \Omega^+_\Lambda} \sigma_0 \, e^{-H_\beta(\sigma)} = \sum_{x \in \mathcal{X}_1} f(x) \, w(x).$$

This rewriting will be particularly powerful if we can use some specific combinatorial properties of the objects in $\mathcal{X}$.

2) HIGH TEMPERATURE EXPANSION AND $\beta_c > 0$

In this section, we will prove the following theorem.

Thm: For every $d \geq 1$, there exists $\beta_0 = \beta_0(d)$ such that

$$\forall \beta < \beta_0(d) \quad \exists c > 0 \quad \text{s.t.} \quad \boxed{\forall n \geq 1 \quad \langle \sigma_0 \rangle^+_{B_n, \beta} \leq e^{-cn}}$$

and $\beta_0(1) = +\infty$

Corollary

- For every $d \geq 2$ $\beta_c(d) > 0$
- $\beta_c(1) = +\infty$.

In order to prove the theorem above, we will use the high temperature expansion (to remember the terminology, remind that this representation is used to prove results for β small (high temperature)).

Let us start by rewriting the partition function.

Fix $\Lambda \subset\subset \mathbb{Z}^d$. We rely on the following elementary identity

$$\forall \varepsilon \in \{\pm 1\} \quad e^{\beta \varepsilon} = \cosh \beta + \varepsilon \sinh \beta$$

$$= \cosh \beta \left(1 + \varepsilon \tanh \beta \right).$$

We have

$$Z^+_{\beta} := \sum_{\sigma \in \Omega^+_\Lambda} \prod_{\{i,j\} \in \overline{E}_\Lambda} e^{\beta \sigma_i \sigma_j}$$

$$= \cosh(\beta)^{|\overline{E}_\Lambda|} \sum_{\sigma \in \Omega^+_\Lambda} \prod_{\{i,j\} \in \overline{E}_\Lambda} (1 + \tanh(\beta)\, \sigma_i \sigma_j)$$

$$= \cosh(\beta)^{|\overline{E}_\Lambda|} \sum_{\sigma \in \Omega^+_\Lambda} \sum_{\eta \subset \overline{E}_\Lambda} \tanh(\beta)^{|\eta|} \prod_{\{i,j\} \in \eta} \sigma_i \sigma_j$$

$$= \cosh(\beta)^{|\overline{E}_\Lambda|} \sum_{\eta \subset \overline{E}_\Lambda} \tanh(\beta)^{|\eta|} \times \sum_{\sigma \in \Omega^+_\Lambda} \prod_{\{i,j\} \in \eta} \sigma_i \sigma_j$$

where, for fixed $\eta \subset \overline{E}_\Lambda$

$$\sum_{\sigma \in \Omega^+_\Lambda} \prod_{\{i,j\} \in \eta} \sigma_i \sigma_j = \sum_{\sigma \in \Omega^+_\Lambda} \prod_{i \in \Lambda} \sigma_i^{|\{j \in \Lambda : \{i,j\} \in \eta\}|}$$

$$= \sum_{\sigma \in \Omega^+_\Lambda} \prod_{i \in \Lambda} \sigma_i^{\mathbb{1}[i \in \partial \eta]}$$

(where we introduced $\partial \eta = \{ i \in \Lambda$ such that $|\{j : \{i,j\} \in \eta\}|$ odd$\}$)

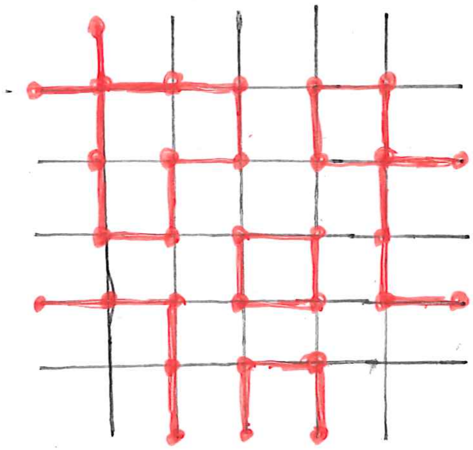
$$= \prod_{i \in \Lambda} \underbrace{\sum_{\sigma_i \in \{\pm 1\}} \sigma_i^{\mathbb{1}[i \in \partial \eta]}}_{= \begin{cases} 0 & \text{if } i \in \partial \eta \\ 2 & \text{if } i \notin \partial \eta. \end{cases}}$$

$$= \begin{cases} 0 & \text{if } \partial \eta \neq \emptyset \\ 2^{|\Lambda|} & \text{if } \partial \eta = \emptyset \end{cases}$$

Finally, we obtain

$$Z^{+}_{\Lambda,\beta} = 2^{|\Lambda|} \cosh(\beta)^{|E_\Lambda|} \sum_{\substack{\gamma \subset \overline{E}_\Lambda \\ \partial\gamma = \emptyset}} \tanh(\beta)^{|\gamma|}$$

An element $\gamma \subset \overline{E}_\Lambda$ with $\partial\gamma = \emptyset$ is a subgraph of $\overline{E}_\Lambda$ where all the "internal degrees" (the degrees of $i \in \Lambda$) are even. We call $\partial\gamma$ the set of <u>sources</u> of γ.



an example of $\gamma \subset \overline{E}_\Lambda$ with $\partial\gamma = \emptyset$

when $\Lambda = \{-2,\dots,2\}^2$

Equivalently, we can prove, for every $A \subset \Lambda \subset\subset \mathbb{Z}^d$

$$\sum_{\sigma \in \Omega^{+}_{\Lambda}} \sigma_A e^{-H^{+}_{\Lambda,\beta}(\sigma)} = 2^{|\Lambda|} \cosh(\beta)^{|E_\Lambda|} \sum_{\substack{\gamma \subset \overline{E}_\Lambda \\ \partial\gamma = A}} \tanh(\beta)^{|\gamma|},$$

which proves the following

<u>Thm</u> (High-temperature expansion)

Fix $\beta \geq 0$. Let $A \subset \Lambda \subset\subset \mathbb{Z}^d$, then

$$\langle \sigma_A \rangle^{+}_{\Lambda,\beta} = \frac{\sum_{\gamma \subset \overline{E}_\Lambda ; \partial\gamma = A} \tanh(\beta)^{|\gamma|}}{\sum_{\gamma \subset \overline{E}_\Lambda ; \partial\gamma = \emptyset} \tanh(\beta)^{|\gamma|}}$$

Notation: Given $\eta \subset E_\Lambda$, we write $G(\eta) = (V(\eta), \eta)$ for the subgraph of $\mathbb{Z}^d$ induced by η, ie $V(\eta) := \bigcup_{\{x,y\} \in \eta} \{x,y\}$.

(on the figure of the previous page, the vertices of $V(\eta)$ are represented by red dots.)

A first application of the high temperature expansion is the computation of $\langle \sigma_0 \rangle^+_{B_m, \beta}$ in dimension $d=1$.

Proposition:

If $d=1$, let $B_m = \{-m, \dots, m\}$, then

$$\langle \sigma_0 \rangle^+_{B_m, \beta} = 2 \, \frac{\tanh(\beta)^{m+1}}{1+\tanh(\beta)^{2m+2}}$$

$$\leq 2 \tanh(\beta)^{m+1}$$

Proof: First there exist only two graphs $\eta \subset E_{B_m}$ such that $\partial\eta = \emptyset$: $\eta = \emptyset$ and $\eta = E_{B_m}$. Hence

$$\sum_{\substack{\eta \subset E_\Lambda \\ \partial\eta = \emptyset}} \tanh(\beta)^{|\eta|} = 1 + \tanh(\beta)^{2m+2}.$$

Also, the graphs illustrated below are the only graphs with $\partial\eta = \{0\}$; $\eta =$  and $\eta =$ 

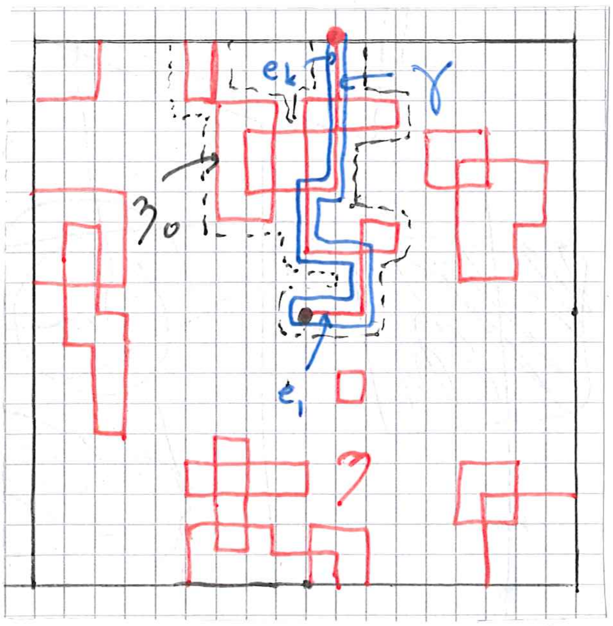
Hence,
$$\sum_{\substack{\eta \subset E_\Lambda \\ \partial\eta = \{0\}}} \tanh(\beta)^{|\eta|} = 2 \tanh(\beta)^{m+1}$$ ∎

Now, we use the high-temperature expansion to prove the theorem stated at the beginning of the section.

<u>Proof</u> (of the existence of $\beta_0(d) > 0$)

The case $d = 1$ and the fact that $\beta_0(1) = +\infty$ follows from the previous proposition. We now focus on more general $d \geq 1$.

The key observation is the following: for every $\eta \subset \overline{E}_{B_m}$ such that $\partial\eta = \{0\}$, there exists a self-avoiding path $\gamma \subset \eta$ connecting 0 to $\mathbb{Z}^d \setminus B_m$ (i.e. $\gamma = \{e_1, \dots, e_k\}$ where $0 \in e_1$, $e_i \cap e_{i+1} \neq \emptyset$, $e_k \in \overline{E}_{B_m} \setminus E_{B_m}$)



← A configuration $\eta \subset \overline{E}_\Lambda$ with $\partial\eta = \{0\}$ and a path $\gamma \subset \eta$ from 0 to $\mathbb{Z}^d \setminus B_m$

To prove this, recall that for every graph $G = (V, E)$

$$\sum_{x \in V} \deg_G(x) = 2|E|.$$

Let η_0 be the set of edges in the connected component of 0 in $G(\eta)$. (see figure in the previous page)

The equality above applied to $G(\eta_0)$ implies

$$\sum_{x \in V(\eta_0)} \deg_{G(\eta_0)}(x) \text{ is even.}$$

Since $\deg_{G(\eta_0)}(0)$ is odd, there must exist another vertex $x \in V(\eta_0) \setminus \{0\}$ s.t. $\deg_{G(\eta_0)}(x)$ is odd. The degree of every vertex $y \in B_n \setminus \{0\}$ being even, we must have $x \in \mathbb{Z}^d \setminus B_n$. Since the graph $G(\eta_0)$ is connected, there exists a path $\gamma \subset \eta_0$ from 0 to x.

In particular, this implies that there exists a path $\gamma \subset \eta$ from 0 of length $|\gamma| = n$. Writing Γ_n for the set of all self-avoiding paths starting at 0 of length n.

We obtain.

$$\sum_{\substack{\eta \subset \overline{E}_{B_n} \\ \partial\eta = \{0\}}} \tanh(\beta)^{|\eta|} \leq \sum_{\gamma \in \Gamma_n} \sum_{\substack{\eta \subset \overline{E}_{B_n} \\ \partial\eta = \{0\} \\ \gamma \subset \eta}} \tanh(\beta)^{|\eta|}$$

$$= \sum_{\gamma \in \Gamma_n} \tanh(\beta)^n \cdot \sum_{\substack{\eta \subset \overline{E}_{B_n} \\ \partial\eta = \{0\} \\ \gamma \subset \eta}} \tanh(\beta)^{|\eta \setminus \gamma|}$$

Now, observe that for fixed $\gamma \in \Gamma_m$, the mapping

$$\{\eta \subset \overline{E_{B_m}} : \partial\eta = \{0\}, \gamma \subset \eta\} \longrightarrow \{\eta \subset \overline{E_{B_m}} : \partial\eta = \emptyset, \gamma \cap \eta = \emptyset\}.$$

$$\eta \longmapsto \eta \setminus \gamma$$

is bijective (its inverse is simply $\eta \mapsto \eta \cup \gamma$).

Hence, we can use a change of variable to show that for fixed γ,

$$\sum_{\substack{\eta \subset \overline{E_{B_m}} : \\ \partial\eta = \{0\} \\ \gamma \subset \eta}} \tanh(\beta)^{|\eta \setminus \gamma|} = \sum_{\substack{\eta' \subset \overline{E_{B_m}} \\ \partial\eta' = \emptyset \\ \gamma \cap \eta' = \emptyset}} \tanh(\beta)^{|\eta'|}$$

$$\leq \sum_{\substack{\eta \subset \overline{E_{B_m}} \\ \partial\eta = \emptyset}} \tanh(\beta)^{|\eta|}.$$

Combine with the previous inequality, we finally get

$$\sum_{\substack{\eta \subset \overline{E_{B_m}} \\ \partial\eta = \{0\}}} \tanh(\beta)^{|\eta|} \leq \sum_{\gamma \in \Gamma_m} \tanh(\beta)^m \cdot \sum_{\substack{\eta \subset \overline{E_{B_m}} \\ \partial\eta = \emptyset}} \tanh(\beta)^{|\eta|}$$

which, by the high-temperature expansion gives

$$\langle \sigma_0 \rangle^+_{B_m, \beta} \leq \sum_{\gamma \in \Gamma_m} \tanh(\beta)^m$$

$$\leq |\Gamma_m| \cdot \tanh(\beta)^m$$

$$\leq (2d)(2d-1)^{m-1} \times \tanh(\beta)^m.$$

This concludes the proof if we choose $(2d-1)\tanh(\beta_0(d)) = 1$ ∎

3) LOW TEMPERATURE EXPANSION AND $\beta_c(d)<\infty$ for $d\geq 2$

Thm: For every $d \geq 2$, we have
$$\beta_c(d) < \infty.$$

Remark: In other words, for β large enough, there is no uniquess of the infinite volume Ising measure. In dimension 2, Aizenman-Higuchi's theorem states that for every $\beta > \beta_c$, all the Ising measures are of the form $\alpha \mu_\beta^+ + (1-\alpha)\mu_\beta^-$, $\alpha \in [0,1]$. This is not the case in higher dimensions, where more general infinite-volume Ising measures exist.

To prove the theorem above, we will apply the Peierls argument to the low temperature representation. In order to give this representation, let us introduce the combinatorial objects needed:

Def: Fix $\Lambda \subset\subset \mathbb{Z}^d$. A subset $\gamma \subset \mathbb{R}^d$ is called a <u>contour in Λ</u> if:

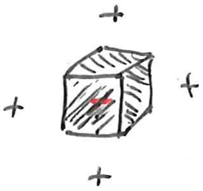
(i) there exists $\sigma \in \Omega_\Lambda^+$ s.t. $\gamma = \partial\Big(\bigcup_{\substack{i\in\mathbb{Z}^d \\ \sigma_i=-1}} i+[-\frac{1}{2},\frac{1}{2}]^d\Big)$

(ii) C is connected.

Notation: if γ is a contour, the configuration σ in (i) is necessarily unique and we can define its size

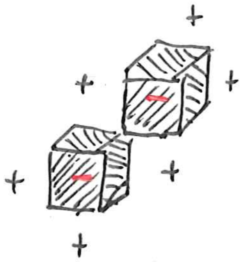
$$|\gamma| := \left| \{ \{i,j\} \in \overline{E}_\Lambda : \sigma_i \sigma_j = -1 \} \right|$$

Ex: in dimension 3.



contour γ associated to $\sigma_i = \begin{cases} -1 & \text{if } i = 0 \\ 1 & \text{if } i \neq 0. \end{cases}$

$|\gamma| = 6$

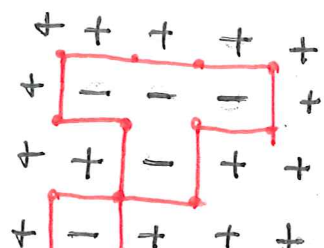


Contour associate to $\sigma_i = \begin{cases} -1 & \text{if } i = 0 \\ -1 & \text{if } i = (1,1,1) \\ 0 & \text{otherwise.} \end{cases}$

$|\gamma| = 12$

Rk: in words, a contour is a connected union of facets (ie. rotated and translated versions of $[0,1]^{d-1} \times \{0\}$) and it size is the number of facets.

Rk: In dimension 2, a contour corresponds to an even* subgraph of $\mathbb{Z}^2$, translated by $\left(\frac{1}{2}, \frac{1}{2}\right)$



(* the degree of every vertex is even).

A contour in $\mathbb{R}^2$

Given $\sigma \in \Omega^+_\Lambda$, we consider

$$\Gamma(\sigma) = \{\gamma_1(\sigma), \dots, \gamma_{m(\sigma)}(\sigma)\}$$

where $\gamma_1(\sigma), \dots \gamma_{m(\sigma)}$ are the connected components (in $\mathbb{R}^d$) of the set

$$\bigcup_{i \in \mathbb{Z}^d : \sigma_i = -1} \left(i + [-\tfrac{1}{2}, \tfrac{1}{2}]^d\right).$$

Note that $\gamma_i(\sigma)$ is a contour for every i.

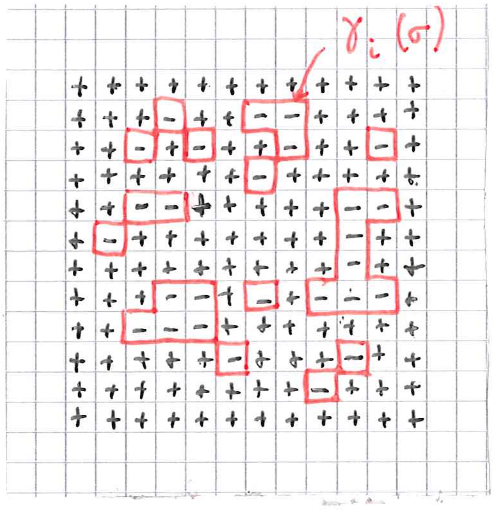


Fig 2: In dimension 2, the contours associated to a configuration σ.

We can easily express the energy of a configuration σ in terms of the size of its contours:

$$H^+_{\Lambda,\beta}(\sigma) = \beta \cdot \left|\{\{i,j\} \in \overline{E}_\Lambda : \sigma_i \neq \sigma_j\}\right| - \beta \left|\{\{i,j\} \in \overline{E}_\Lambda : \sigma_i = \sigma_j\}\right|$$

$$= 2\beta \cdot \left|\{\{i,j\} \in \overline{E}_\Lambda : \sigma_i \neq \sigma_j\}\right| - 2\beta |\overline{E}_\Lambda|$$

$$= 2\beta\left(\sum_{\gamma \in \Gamma(\sigma)} |\gamma|\right) - 2\beta |\overline{E}_\Lambda|$$

Now, let $\mathcal{C}_\Lambda = \{\Gamma(\sigma), \sigma \in \Omega^+_\Lambda\}$ be the set of all families of disjoint contours in Λ. Since

$$\sigma \longmapsto \Gamma(\sigma) \quad .$$

is a bijection from Ω^+_Λ to $\mathcal{C}$, we can write

$$Z^+_\Lambda = \sum_{\sigma \in \Omega^+_\Lambda} e^{-H^+_{\Lambda,\beta}(\sigma)}$$

$$= e^{2\beta|E_\Lambda|} \times \sum_{\sigma \in \Omega^+_\Lambda} \prod_{\gamma \in \Gamma(\sigma)} e^{-2\beta|\gamma|}$$

$$= e^{2\beta|E_\Lambda|} \sum_{\Gamma \in \mathcal{C}} \prod_{\gamma \in \Gamma} e^{-2\beta|\gamma|}$$

Now observe that $\sigma_0 = -1$ if and only if the number of contours in $\Gamma(\sigma_0)$ surrounding* 0 is odd. (* formally we can define the number of contours surrounding 0 in Λ as the minimal number of intersections of $\bigcup_{\{x,y\} \in \pi} [x,y]$ with $\bigcup_{\gamma \in \Gamma} \gamma$, when π runs over all paths in $\mathbb{Z}^d$ from 0 to $\mathbb{Z}^d \setminus \Lambda$.). Writing

$$\mathcal{C}^0_\Lambda = \{\Gamma \in \mathcal{C}_\Lambda \text{ with odd numbers of contours surrounding } 0\},$$

we obtain

$$\boxed{\mu^+_{\Lambda,\beta}(\sigma_0 = -1) = \frac{\sum_{\Gamma \in \mathcal{C}^0_\Lambda} \prod_{\gamma \in \Gamma} e^{-2\beta|\gamma|}}{\sum_{\Gamma \in \mathcal{C}_\Lambda} \prod_{\gamma \in \Gamma} e^{-2\beta|\gamma|}}}$$

We are now ready to prove the theorem:

Proof of the theorem.

We prove only the case $d = 2$, where the proof is slightly easier since contours are subgraphs of the dual lattice. In higher dimension, the theorem can be proven similarly. Alternatively, the case of $d \geq 3$ can be deduced from the case $d = 2$ by using that $\beta_c(d)$ is decreasing in d, and in particular,

$$\beta_c(d) \leq \beta_c(2).$$

This can be proved using inequality (A 2) in [1].

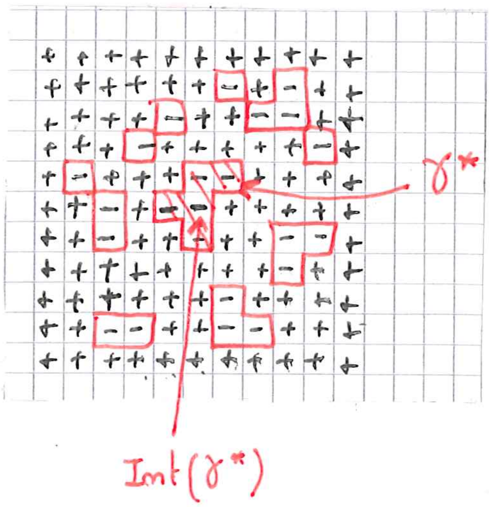
Let $B_n = \{-n, \dots, n\}^2$. We will prove that there exists $\beta > 0$ large enough and a constant $c > 0$ such that for every n,

$$\mu^{+}_{B_n,\beta}(\sigma_0 = -1) \leq \frac{1}{2} - c .$$

This directly implies that $\langle \sigma_0 \rangle^{+}_{\beta} = \lim_{n\to\infty} \left(1 - 2\mu^{+}_{B_n,\beta}(\sigma_0 = -1)\right) \geq 2c > 0$

If $\Gamma \in \mathcal{E}^{0}_{B_n}$, there must exist at least one contour surrounding 0.

Let $\gamma^*(\Gamma)$ be the first contour of Γ surrounding 0.



Then $\displaystyle\sum_{\Gamma \in \mathcal{E}^{0}_{B_n}} \prod_{\gamma \in \Gamma} e^{-2|\gamma|\beta}$

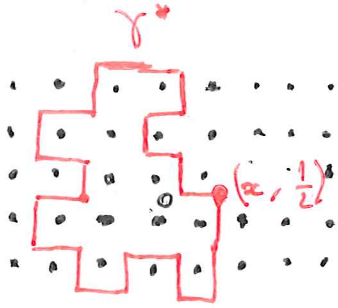
$$= \sum_{\gamma^*} e^{-2|\gamma^*|\beta} \underbrace{\sum_{\Gamma : \gamma^*(\Gamma) = \gamma^*} \prod_{\gamma \in \Gamma \setminus \{\gamma^*\}} e^{-2|\gamma|\beta}}_{\leq \sum_{\Gamma \in \mathcal{E}_{B_n} \setminus \mathrm{Int}(\gamma^*)} \prod_{\gamma \in \Gamma} e^{-2|\gamma|\beta} \;\leq\; \sum_{\Gamma \in \mathcal{E}_{B_n}} \prod_{\gamma \in \Gamma} e^{-2|\gamma|\beta} .}$$

Therefore,

$$\mu^{+}_{B_m,\beta}[\sigma_0 = -1] \leq \sum_{\gamma^*} e^{-2|\gamma^*|\beta}$$

where the sum is over all the contours surrounding 0

Now fix $k \geq 4$. Consider a contour γ^* surrounding 0, of size $|\gamma^*| = k$. Then there must exist a point $x \in \frac{1}{2} + \{0, \dots, k-1\}$ s.t. $(x, \frac{1}{2}) \in \gamma^*$, and the contour



γ^* can be seen as a edge-self-avoiding path starting at $(x, \frac{1}{2})$, of length k.

Therefore the number N_k of contours surrounding 0, of size k, satisfies

$$N_k \leq \underbrace{k}_{\text{"choice of } x\text{"}} \times \underbrace{4 \cdot 3^{k-1}}_{\text{"bound on the number of paths from } (x, \frac{1}{2})\text{"}}$$

Finally

$$\mu^{+}_{B_m,\beta}[\sigma_0 = -1] = \sum_{k \geq 4} N_k \, e^{-2k\beta}$$

$$\leq \sum_{k \geq 4} k \times 4 \times 3^{k-1} e^{-2k\beta}.$$

$$\leq \frac{1}{2} - c \qquad \text{if } \beta \text{ large enough}$$

4) DIMENSION 2: KRAMER-WANNIER DUALITY.

In this section, we focus on the dimension $d=2$. In this particular case, we use tools from planar graph theory to prove a correspondance between:

- the low temperature expansion of an Ising model on $\mathbb{Z}^2$ at inverse-temperature β.
- the high temperature expansion of an Ising model on the dual $(\mathbb{Z}^2)^*$ at inverse temperature β^*.

This correspondance, known as Kramer-Wannier duality will allow us to prove the following theorem. We are still working without external field, and we simply write $f(\beta) = f(\beta, 0)$ for the pressure at inverse-temperature β and external field $h=0$.

Thm:

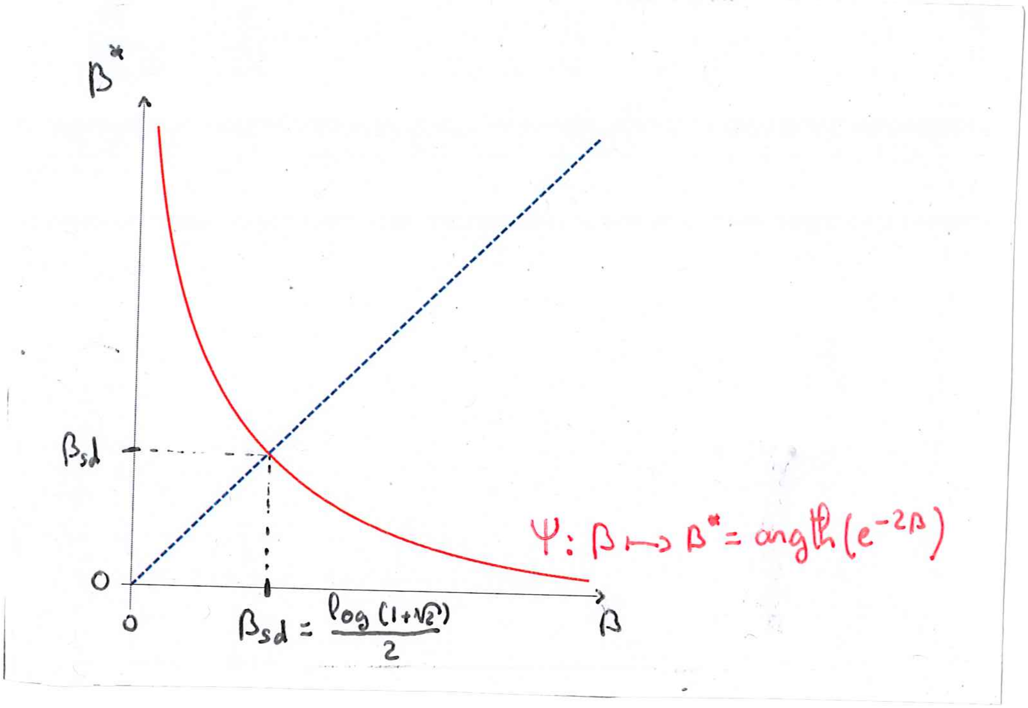
Let $\beta > 0$ and $\beta^* := \operatorname{argth}(e^{-2\beta})$. Then, the pressure satisfies

$$\boxed{f(\beta) = f(\beta^*) - \log(\sinh(2\beta^*))}$$

Remark. The function $\Psi: \left| \begin{array}{ccc} \mathbb{R}_+\setminus\{0\} & \longrightarrow & \mathbb{R}_+\setminus\{0\} \\ \beta & \longmapsto & \beta^* := \operatorname{argth}(e^{-2\beta}) \end{array} \right.$ satisfies.

- Ψ has a unique fixed point $\beta_{sd} := \frac{1}{2}\log(1+\sqrt{2})$
- $\Psi \circ \Psi = \mathrm{Id}$
- $\Psi((0, \beta_{sd}]) = [\beta_{sd}, \infty)$

(sd stems for "self-dual")

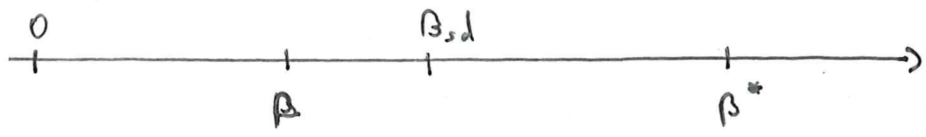


Graph of the function. $\Psi : \beta \longrightarrow \beta^* = \operatorname{argth}(e^{-2\beta})$

In particular, the previous theorem implies for every $\beta > 0$

$$(f \text{ is differentiable at } \beta) \iff (f \text{ is differentiable at } \beta^*)$$

If we assume that β_c is the unique point of $\mathbb{R}_+ \setminus \{0\}$ where f is not differentiable, then we must have $\beta_c = \beta_{sd}$ (otherwise β_c and β_c^* are two non differentiability points of f).



This argument is of course heuristic because we make a strong assumption. Nevertheless, the fact that $\beta_c = \beta_{sd}$ is a correct guessing and the following result will be established later using the FK-representation.

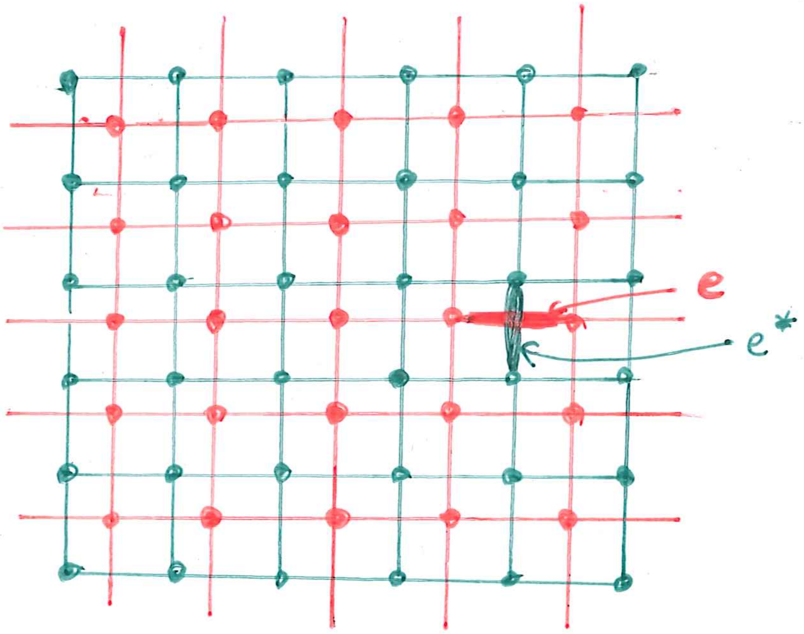
Thm: In dim. 2, we have

$$\boxed{\beta_c(2) = \beta_{sd} = \frac{1}{2}\log(1+\sqrt{2})}$$

In order to prove the first theorem, we will consider representations of the Ising model on two graphs:

- the primal graph $(B_m, \overline{E_{B_m}})$ [a subgraph of $\mathbb{Z}^2$]
- its dual graph $(B_m^*, E_{B_m^*})$ [a subgraph of $(\mathbb{Z}^2)^* = (\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$]

where $B_m = \{-m, \dots, m\}^2$ and $B_m^* = \{-m-\frac{1}{2}, \dots, m+\frac{1}{2}\}^2$



In red: the set B_m and the edges $\overline{E_{B_m}}$

In green: the set B_m^* and the edges $E_{B_m^*}$.

Note that we can define a bijection

$$\begin{aligned} \overline{E_{B_m}} &\longrightarrow E_{B_m^*} \\ e &\longmapsto e^* \quad (\text{"the unique edge of } E_{B_m^*} \text{ crossed by } e\text{"}). \end{aligned}$$

The Kramers - Wannier duality relies on the correspondance between:

- spin configurations on the primal graph $(B_n, \overline{E}_{B_n})$
- Eulerian subgraphs on the dual graph $(B_n^*, E_{B_n^*})$

Definition: Let $G = (V, E)$ be a graph. Let $\eta \subset E$.

We say that η is an Eulerian subgraph of G if, for every $v \in V$, the number of edges of η, $\deg_\eta(v)$ adjacent to η is even.

Notation: We write $\mathcal{E}_n^*$ for the set of all Eulerian subgraphs of $(B_n^*, E_{B_n^*})$

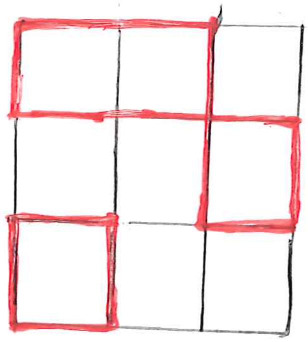


Illustration of an Eulerian subgraph.

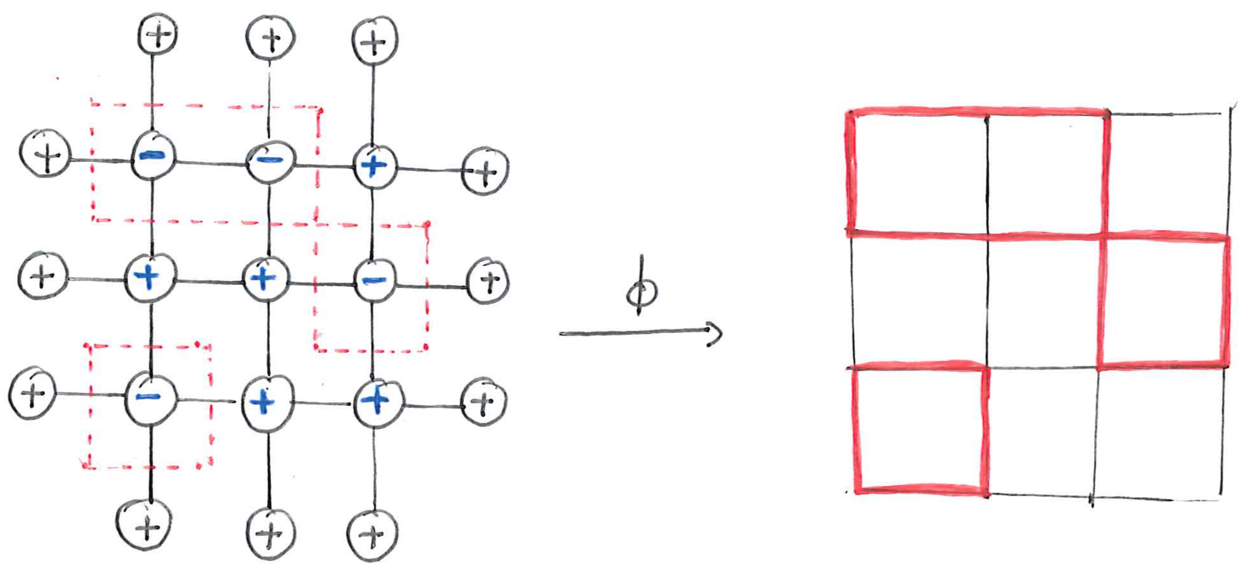
Proposition:

For every $\sigma \in \Omega_{B_n}^+$, the set of edges

$$\eta^* := \{ \{x,y\}^*, \{x,y\} \in \overline{E}_{B_n}, \sigma_x \sigma_y = -1 \}$$

is an Eulerian subgraph of $(B_n^*, E_{B_n^*})$, and

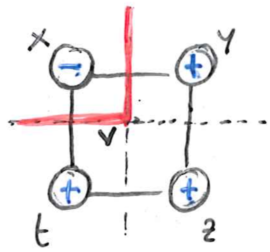
$$\phi : \Omega_{B_n}^+ \longrightarrow \mathcal{E}_n^* \quad \text{is a bijection.}$$
$$\sigma \longmapsto \eta^*$$



$\sigma \in \Omega^+_{B_m}$ $\qquad\qquad$ $\gamma^*(\sigma) \in E_{B_m^*}$

Proof: We first prove that $\gamma^* \in \mathcal{E}^*_m$.

Let $v \in B_m^*$, let x, y, z, t be the 4 vertices of $\mathbb{Z}^2$ surrounding v.



We have $(-1)^{\deg_{\gamma^*}(v)}$

$$= (\sigma_x \sigma_y).(\sigma_y \sigma_z).(\sigma_z \sigma_t).(\sigma_t \sigma_x)$$

$$= \sigma_x^2 \sigma_y^2 \sigma_z^2 \sigma_t^2 = +1$$

Hence $\deg_{\gamma^*}(v)$ is even.

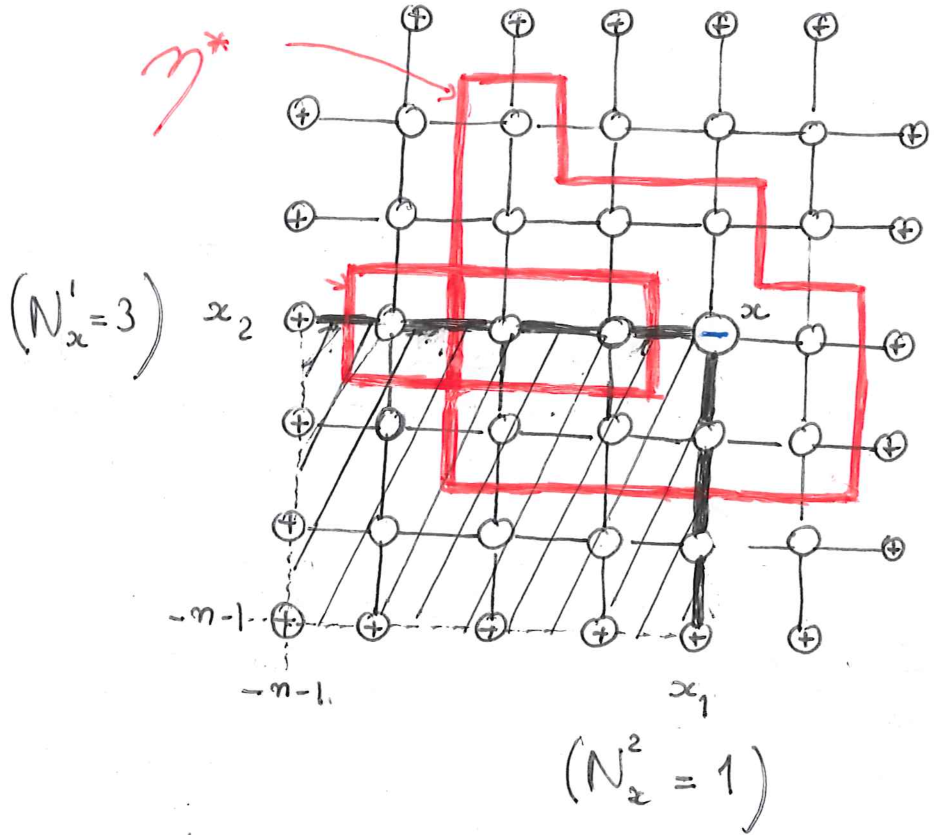
ϕ is injective

Let $\sigma, \tau \in \Omega^+_{B_m}$ s.t $\sigma \neq \tau$. Since σ and τ coincide outside B_m, there must exist $\{x, y\} \in \overline{E_{B_m}}$ such that $\sigma_x = \tau_x$ and $\sigma_y \neq \tau_y$, which implies $\phi(\sigma) \neq \phi(\tau)$.

ϕ is surjective:

Let $\eta^* \in \mathcal{E}_n^*$ be a fixed Eulerian subgraph of $(B_n^*, E_{B_n^*})$

Let $x = (x_1, x_2) \in B_n$



Let N_x^1 (resp. N_x^2) be the number of edges of η^* crossing the horizontal line $[-n-1, x_1] \times \{x_2\}$ (resp. the vertical line $\{x_1\} \times [-n-1, x_2]$).

<u>Claim:</u> N_x^1 and N_x^2 have the same parity.

We assume this claim for the moment. It allows us to define, for every $x \in B_n$

$$\sigma_x = (-1)^{N_x^1} = (-1)^{N_x^2}$$

Using the expression $\sigma_x = (-1)^{N_x^1}$, we get

$$\forall \text{ horizontal edge } \{x,y\} \in \overline{E}_{B_m} \quad \left(\sigma_x \sigma_y = -1\right) \Leftrightarrow \left(\{x,y\}^* \in \gamma^*\right)$$

and we get the equivalent statement for the vertical edges using the expression $\sigma_x = (-1)^{N_x^2}$.

This proves $\phi(\sigma) = \gamma^*$, as desired.

In order to prove the claim, we consider the rectangle $R = [-n-1, x_1] \times [-n-1, x_2]$ (see the hatched region in the picture above)

Summing $\deg_{\gamma^*}(v)$ over all vertices $v \in B_m^* \cap R$ we obtain

$$\sum_{v \in B_m^* \cap R} \deg_{\gamma^*}(v) = 2 \cdot \left|\{\text{edges of } \gamma^* \text{ in } R\}\right| + N_x^1 + N_x^2$$

(the internal edges are counted two times, and $N_x^1 + N_x^2$ counts the edges of γ^* at the boundary of R)

Hence $N_x^1 + N_x^2$ is even, which concludes the claim, and ends the proof. $\square$

Remark:

By the proposition above, we can compute the number of Eulerian subgraphs of $(B_m^*, E_{B_m^*})$ as

$$|\mathcal{E}_m^*| = |\Omega_{B_m}^+| = 2^{|B_m|} = 2^{(m+1)^2}.$$

More generally, it is possible to show that the number of Eulerian subgraphs of a given connected graph $G=(V,E)$ is equal to $2^{|E|-|V|+1}$ (see e.g. [Diesel, Graph theory]).

Low-temperature expansion in $d=2$

We have for every $\sigma \in \Omega_{B_m}^+$

$$H_{B_m,\beta}^+(\sigma) = 2\beta \underbrace{|\{\{x,y\} \in \overline{E}_{B_m} : \sigma_x \neq \sigma_y\}|}_{=|\phi(\sigma)|} - \beta |\overline{E}_{B_m}|$$

Therefore,

$$\boxed{Z_{B_m,\beta}^+} = \sum_{\sigma \in \Omega_{B_m}^+} e^{H_{B_m,\beta}^+(\sigma)} = e^{\beta|\overline{E}_{B_m}|} \sum_{\sigma \in \Omega_{B_m}^+} e^{-2\beta|\phi(\sigma)|}$$

$$\overset{\text{Prop.}}{=} \boxed{e^{\beta|\overline{E}_{B_m}|} \sum_{\gamma^* \in \mathcal{E}_m^*} e^{-2\beta|\gamma^*|}}$$

We are now in a position to prove the theorem at the beginning of the section (relating $f(\beta)$ and $f(\beta^*)$).

<u>Proof of the Thm:</u>

High-temperature expansion on $(B_n^*, E_{B_n^*})$ at β^*:

$$Z^{\emptyset}_{B_n^*,\beta^*} = 2^{|B_n^*|} \cosh(\beta^*)^{|E_{B_n^*}|} \sum_{\gamma^* \in \mathcal{E}_n^*} \tanh(\beta^*)^{|\gamma^*|}$$

ie: $$\sum_{\gamma^* \in \mathcal{E}_n^*} \tanh(\beta^*)^{|\gamma^*|} = Z^{\emptyset}_{B_n^*,\beta^*}\, 2^{-|B_n^*|} \cosh(\beta^*)^{-|E_{B_n^*}|}$$

Low-temperature expansion on $(B_n, \overline{E}_{B_n})$ at β:

$$\sum_{\gamma^* \in \mathcal{E}_n^*} e^{-2\beta|\gamma^*|} = e^{-\beta|\overline{E}_{B_n}|} Z^{+}_{B_n,\beta}$$

Since $\tanh(\beta^*) = e^{-2\beta}$, we obtain

$$e^{-2\beta\left(\frac{|\overline{E}_{B_n}|}{2}\right)} Z^{+}_{B_n,\beta} = Z^{\emptyset}_{B_n,\beta}\, 2^{-|B_n^*|} \cosh(\beta^*)^{-2\left(\frac{|E_{B_n^*}|}{2}\right)}$$

taking the log and dividing by $|B_n|$ gives

$$-\frac{|\overline{E}_{B_n}|}{2|B_n|}\log(e^{+2\beta}) + f^{+}_{B_n}(\beta) = f^{\emptyset}_{B_n^*}(\beta) - \frac{|B_n^*|}{|B_n|}\log 2$$
$$- \frac{|E_{B_n^*}|}{2|B_n|}\log(\cosh(\beta^*)^2).$$

taking the limit as n tends to infinity, we finally obtain

$$f(\beta) = f(\beta^*) + \log\left(\frac{e^{2\beta}}{2\cosh(\beta^*)^2}\right)$$

$$= f(\beta^*) - \log(\sinh(2\beta^*)) \quad \blacksquare$$

STOCHASTIC DOMINATION

Let Ω be a Polish space, equipped with a partial ordering $\leq$.
and its Borel σ-algebra.

1 Definition and first examples.

<u>Def</u>: Let μ, ν be two probability measures on Ω.
We say that μ is <u>stochastically dominated</u> by ν (written $\mu \ll \nu$)
if for every $f : \Omega \longrightarrow \mathbb{R}$ increasing measurable bounded

$$\int f \, d\mu \leq \int f \, d\nu .$$

<u>Exple 1</u>

$\Omega = \mathbb{R}$. For $x > 0$ consider μ_x the law of a uniform random variable on $[0, x]$ (ie $d\mu_x = \mathbb{1}_{[0,x]} \times \frac{1}{x} \, dt$)

Then $0 < x \leq y \Rightarrow \mu_x \ll \mu_y$

<u>Proof</u>: Let X be a uniform r.v. on $[0, x]$

Then $Y = \frac{y}{x} \cdot X$ is a uniform r.v on $[0, y]$

since almost surely $X \leq Y$

we have for every $f \nearrow$ measurable bounded

$$f(X) \leq f(Y) \text{ a.s.}$$

Therefore, by taking the expectation, we obtain

$$\underbrace{\mathbb{E}[f(X)]}_{\int f \, d\mu_x} \leq \underbrace{\mathbb{E}[f(Y)]}_{\int f \, d\mu_y}$$

Exple 2:

$\Omega = \{0,1\}$. For $p \in [0,1]$, let $\mu_p = \text{Bernoulli}(p)$.

Then $0 \leq p \leq q \leq 1 \Rightarrow \mu_p \ll \mu_q$

Proof:

Let U be a uniform random variable in $[0,1]$.

Define $X = \begin{cases} 1 & \text{if } U \leq p \\ 0 & \text{if } U > p \end{cases}$ and $Y = \begin{cases} 1 & \text{if } U \leq q \\ 0 & \text{if } U > q \end{cases}$

Then $X \sim \mu_p$ and $Y \sim \mu_q$

$$p < q \Rightarrow X \leq Y \text{ a.s.} \Rightarrow \forall f : \Omega \to \mathbb{R} \nearrow \quad f(X) \leq f(Y)$$

$$\Rightarrow \underbrace{\mathbb{E}[f(X)]}_{\int f \, d\mu_p} \leq \underbrace{\mathbb{E}[f(Y)]}_{\int f \, d\mu_q}. \qquad \square$$

In the two examples above, we relied on a coupling method to show the stochastic domination.

Def: Let (E,μ), (F,ν) be two probability space. We call coupling of μ and ν a probability measure P on the product space $E \times F$ such that

- its first marginal is μ ($P[A \times F] = \mu(A)$ $\forall A$ measurable)
- its second marginal is ν ($P(E \times B) = \nu(B)$ $\forall B$ measurable)

In the two examples above we prove $\mu \ll \nu$ by constructing a coupling P (on $\Omega \times \Omega$) of μ and ν such that

$$P\left(\{(\omega,\eta) \in \Omega \times \Omega : \omega \leq \eta\} \right) = 1$$

(P is the law of the pair (X,Y))

This easily implies the desired stochastic domination. For all the applications in this course, we will always prove stochastic domination by constructing a coupling of the two measures. Actually, Strassen's theorem states that the reciprocal is also true if Ω is a Polish space.

Theorem: Assume that Ω is Polish. Let μ, ν be two probability measures on Ω. The following are equivalent:

(i) $\mu \ll \nu$

(ii) there exists a coupling P of μ and ν such that

$$P\left(\{(\omega,\eta) \in \Omega \times \Omega : \omega \leq \eta\} \right) = 1$$

(iii) there exist two random variables $X \sim \mu$ and $Y \sim \nu$ on the same probability space and such that

$$X \leq Y \quad \text{a.s.}$$

Proof:

(ii) $\Rightarrow$ (iii) consider a random variable (X,Y) on $\Omega \times \Omega$ with law P.

(iii) $\Rightarrow$ (i) if $X \leq Y$ a.s.
then for every $f : \Omega \to \mathbb{R}$ increasing measurable bounded, we have

$$f(X) \leq f(Y) \quad \text{a.s.}$$

taking the expectation gives $\int f d\mu \leq \int f d\nu$.

(i) $\Rightarrow$ (ii) see e.g. Lindvall '99 (e.c.p) or Werner (percolation et modèle d'Ising p. 98) for the case Ω finite.

2. STOCHASTIC DOMINATION ON PRODUCT SPACES.

In this section, we fix a finite set S, and consider $\Omega = \{0,1\}^S$ equipped with the product ordering $\leq$.

$$\eta \leq \psi \iff \forall i \in S \quad \eta_i \leq \psi_i .$$

Exercice:

For $p \in [0,1)$ let $\mu_p = \text{Bernoulli}(p)^{\otimes S}$ be the law of $X = (X_s)_{s\in S}$ where X_s are iid Bernoulli variables with parameter p.

Prove that $p \leq q \iff \mu_p \ll \mu_q$...

For non-product measures (which correspond to random variables $X = (X_s)_{s \in S}$ with dependencies, X_s may depend on X_t for $s \neq t$), it is a priori not obvious to prove stochastic domination. We present here the Holley criterion which is a powerful tool in order to prove stochastic dominations on Ω.

Thm: (Holley criterion).

Let μ, ν be two positive measures on Ω (i.e. $\mu(\eta), \nu(\eta) > 0$ for every $\eta \in \Omega$). Assume that for every $\eta \leq \psi$

$$\frac{\mu[\eta^e]}{\mu[\eta_e]} \leq \frac{\nu[\psi^e]}{\nu[\psi_e]}$$

Then $\mu \ll \nu$.

Préliminary:

The proof of the theorem is based on a Markov chain method. In order to construct a suitable coupling for μ and ν, we will couple two Markov Chains $X = (X_n)_n$ and $Y = (Y_n)_n$ with respective invariant measures μ and ν, s.t. $X_n \leq Y_n$ for every n.

Before that, let use describe one Markov chain $(X_n)_n$ with invariant measure μ.

The chain starts from a fixed configuration $X_0 = \eta_0$. Then for $n \geq 0$ X_{n+1} is constructed from X_n as follows.

Pick $S_n \in S$ uniformly at random.

Define . for $s \neq S_n$ $X_{n+1}(s) = X_n(s)$

. for $s = S_n$

$$X_{n+1}(s) = \begin{cases} 1 & \text{with prob. } \mu[\omega(s)=1 \mid \forall t \neq s \ \omega(t) = X_n(t)] \\ 0 & \text{otherwise} \end{cases}$$

One can check that (X_n) is an irreducible aperiodic Markov chain on Ω with invariant measure μ.

In particular $E[f(X_n)] \xrightarrow[n \to \infty]{} \int f \, d\mu$

Proof of Thm:

Let $\eta \leq \psi$ be two configurations in Ω, let $s \in S$.

$$\mu[\omega(s)=1 \mid \forall t \in S\setminus\{s\}\ \omega(t)=\eta(t)]$$

$$= \frac{\mu(\eta^s)}{\mu(\eta_s)+\mu(\eta^s)} = \frac{1}{1+\mu(\eta_s)/\mu(\eta^s)}$$

$$\leq \frac{1}{1+\upsilon(\psi_s)/\upsilon(\psi^s)} = \upsilon[\omega(s)=1 \mid \forall t\in S\setminus\{s\}\ \omega(t)=\psi(t)] \qquad (*)$$

Let $S_1, \dots, S_n, \dots$ be an iid sequence of uniform random variables in S : for every $s \in S$ $\mathbb{P}[S_i=s] = \frac{1}{|S|}$.
Let $U_1, \dots, U_n, \dots$ be an iid sequence of uniform random variables in $[0,1]$.

We construct a Markov Chain (X_n, Y_n) on $\Omega\times\Omega$ as follows. Fix a configuration $\eta_0 \in \Omega$ and set

$$(X_0, Y_0) = (\eta_0, \eta_0).$$

For $n \geq 0$, define (X_{n+1}, Y_{n+1}) as follows:

For $s \neq S_{n+1}$, set $X_{n+1}(s) = X_n(s)$ and $Y_{n+1}(s) = Y_n(s)$.

For $s = S_{n+1}$ set

$$X_{n+1}(s) = \begin{cases} 1 & \text{if } U_{n+1} \leq \mu[\omega(s)=1 \mid \forall t\neq s\ \omega(t)=X_n(s)] \\ 0 & \text{otherwise} \end{cases}$$

and

$$Y_{n+1}(s) = \begin{cases} 1 & \text{if } U_{n+1} \leq \upsilon[\omega(s)=1 \mid \forall t\neq s\ \omega(t)=Y_n(s)] \\ 0 & \text{otherwise} \end{cases}$$

Using (*), we have by induction that

$$X_n \leq Y_n \quad \text{for every } n \geq 1.$$

Therefore, for every $f : \Omega \to \mathbb{R}$ increasing

$$\mathbb{E}[f(X_n)] \leq \mathbb{E}[f(Y_n)]$$

One can check that

- X_n is an irreducible Markov Chain on Ω (because μ is positive, it is possible to move from one configuration to an other by replacing bits by bits all the places where the configurations differ)

- μ is an invariant measure (because it is reversible: for every η and every $s \in S$.

$$\mu(\eta^s) \cdot \mathbb{P}[X_{n+1} = \eta_s \mid X_n = \eta^s]$$

$$= \mu(\eta^s) \cdot \frac{1}{|S|} \cdot \frac{\mu(\eta_s)}{\mu(\eta_s) + \mu(\eta^s)}$$

$$= \mathbb{P}[X_{n+1} = \eta^s \mid X_n = \eta_s)\ \mu(\eta_s) \Big)$$

Hence $\mathbb{E}[f(X_n)] \xrightarrow[n\to\infty]{} \int f \, d\mu$.

And equivalently $\mathbb{E}[f(Y_n)] \longrightarrow \int f \, d\nu$ $\blacksquare$

<u>Rk</u>: In the proof above, it is actually possible to construct a coupling P of μ and ν s.t. $P[\{(\omega,\eta) : \omega \leq \eta\}] = 1$ by considering an invariant measure of the Markov chain (X_n, Y_n) on $\Omega \times \Omega$. (Exercice).

<u>Exercice</u>:

Let $\Lambda \subset\subset \mathbb{Z}^d$, $\xi, \psi \in \{0,1\}^{\mathbb{Z}^d}$ b.c.

Prove using Holley-Criterion that

$$\xi \leq \psi \Rightarrow \mu^{\xi}_{\Lambda,\beta,h} \ll \mu^{\psi}_{\Lambda,\beta,h}$$

$$h \leq h' \Rightarrow \mu^{\rho}_{\Lambda,\beta,h} \ll \mu^{\rho}_{\Lambda,\beta,h'}.$$

(Above we see the Ising measures as measures on $\Omega = \{0,1\}^{\Lambda}$, via the identification $\Omega \overset{bij}{\simeq} \Omega^{\xi}_{\Lambda} \overset{bij}{\simeq} \Omega^{\psi}_{\Lambda}$)

<u>FKG INEQUALITY</u>.

In this section, we give a criterion, based on the Holley criterion, that allows one to prove FKG inequality for dependent measures.

<u>Thm</u>: Let μ be a <u>positive</u> measure on $\Omega = \{0,1\}^S$ s.t.

$$\forall s \in S \quad \forall \eta \leq \psi \quad \frac{\mu(\eta^s)}{\mu(\eta_s)} \leq \frac{\mu(\psi^s)}{\mu(\psi_s)}$$

Then μ satisfies the FKG-inequality:

$$\forall f,g : \Omega \to \mathbb{R} \text{ increasing} \quad \int fg \, d\mu \geq \int f d\mu \cdot \int g \, d\mu$$

Proof: Without loss of generality we can assume that $f(\omega) > 0 \quad \forall \omega \in \Omega$ (if not, consider $f + c$, where c is a large constant).

Consider the positive probability measure ν defined by

$$\forall \psi \in \Omega \quad \nu(\psi) := \frac{1}{\int f \, d\mu} \cdot f(\psi) \cdot \mu(\psi)$$

Since f is increasing we have, for every $\psi \geq \eta$

$$\frac{\nu(\psi^e)}{\nu(\psi_e)} = \underbrace{\frac{f(\psi^e)}{f(\psi_e)}}_{\geq 1} \cdot \frac{\mu(\psi^e)}{\mu(\psi_e)} \geq \frac{\mu(\eta^e)}{\mu(\eta_e)}$$

Therefore, by Holley criterion, $\mu \ll \nu$ and

$$\int g \, d\mu \leq \int g \, d\nu = \frac{1}{\int f \, d\mu} \int f g \, d\mu$$

which concludes the proof ■

Application: proof of the FKG - inequality for the Ising measure.

Let $\Lambda \subset\subset \mathbb{Z}^d$ and $\Omega_\Lambda = \{0,1\}^\Lambda$. $\mu = \mu^{\emptyset}_{\Lambda,\beta,h}$.

For $\sigma \in \Omega_\Lambda$ we have for $i \in \Lambda$

$$H^{\emptyset}_{\Lambda,\beta,h}(\sigma^i) - H^{\emptyset}_{\Lambda,\beta,h}(\sigma_i) = -\beta \sum_{j : \{i,j\} \in E_\Lambda} (\sigma^i - \sigma_i) \cdot \sigma_j - h(\sigma^i - \sigma_i)$$

$$= -2\beta \sum_{j : \{i,j\} \in E_\Lambda} \sigma_j - 2h$$

Hence, for every $\sigma \leq \tilde{\sigma}$

$$H^{\emptyset}_{\Lambda,\beta,h}(\sigma^{i}) - H^{\emptyset}_{\Lambda,\beta,h}(\sigma_{i}) \geq H^{\emptyset}_{\Lambda,\beta,h}(\tilde{\sigma}^{i}) - H^{\emptyset}_{\Lambda,\beta,h}(\tilde{\sigma}_{i})$$

ie
$$\frac{\mu^{\emptyset}_{\Lambda,\beta,h}(\sigma^{i})}{\mu^{\emptyset}_{\Lambda,\beta,h}(\sigma_{i})} \leq \frac{\mu^{\emptyset}_{\Lambda,\beta,h}(\tilde{\sigma}^{i})}{\mu^{\emptyset}_{\Lambda,\beta,h}(\tilde{\sigma}_{i})}.$$